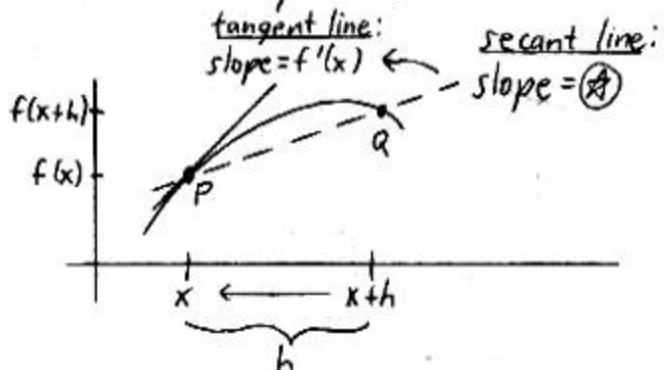


Larson:
1730-60Euler, Jean
D'Alembert16.3: PARTIAL DERIVATIVES (PDS)(A) Review Calc I: $y = f(x)$ 

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

or fix $x = x_0$

Δ

(B) Now, Calc III: $z = f(x, y)$

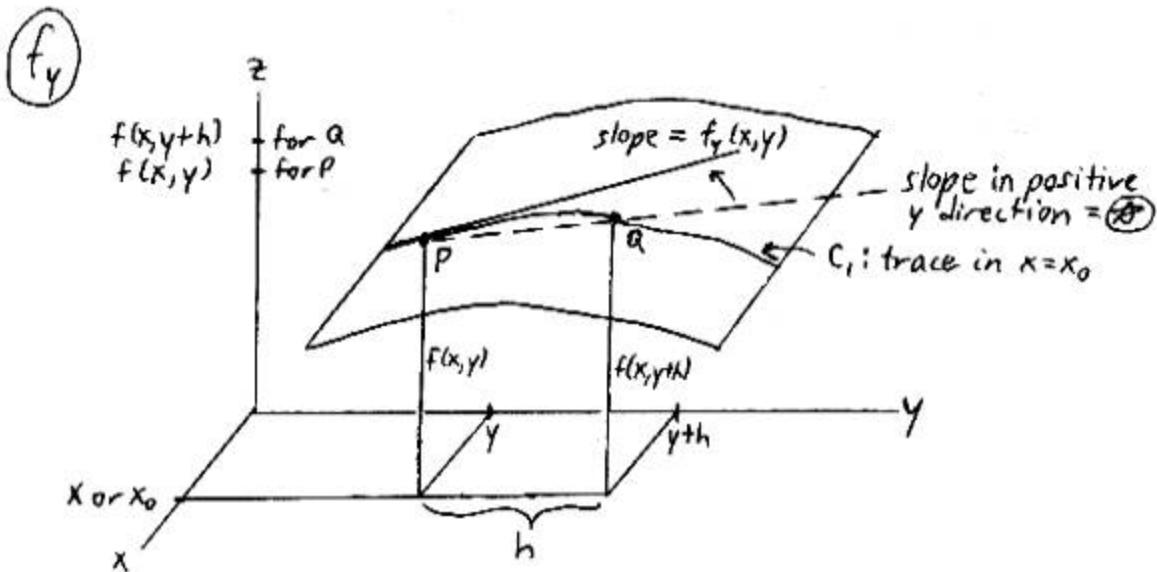
Δ, δ

$$f_x = \frac{\partial f}{\partial x} = \begin{matrix} \text{"del"} \\ \downarrow \end{matrix} \text{ the partial derivative of } f \text{ with respect to } x$$

$$f_y = \frac{\partial f}{\partial y} = \begin{matrix} \text{"wrt"} \\ \downarrow \end{matrix} y$$

Leibniz notation

He had d, not ∂.



$$f_y(x, y) = \lim_{h \rightarrow 0} \frac{f(x, y+h) - f(x, y)}{h}$$

\textcircled{A}

= slope of tangent line to C_1 at P

= instantaneous rate of change of f
wrt y at P

We treat x as constant, and we differentiate $f(x, y)$ wrt y .

f_x

$$f_x(x, y) = \lim_{h \rightarrow 0} \frac{f(x+h, y) - f(x, y)}{h}$$

We treat y as constant, and we differentiate $f(x, y)$ wrt x .

(C) Exs

D_x rules from Calc I extend naturally.

Chain Rule Ex

Books omit ()

How to Ace →

$$\underbrace{\frac{\partial}{\partial x}}_{D_x} (\sin u) = (\cos u) \underbrace{\left(\frac{\partial u}{\partial x}\right)}_{D_x}$$

Ex $f(x,y) = xy^3 + \ln(2x - 3y^2)$ $x \overset{x}{\cancel{y}}$

a) Find $f_x(x,y)$

$$f(x,y) = \underbrace{xy^3}_{\text{treat as constant}} + \underbrace{\ln(2x - 3y^2)}_{\#}$$

← Would ln properties help?
Not here...

Calc I: $D(\ln A) = \frac{1}{A} \cdot D(A)$

$$f_x(x,y) = \underbrace{y^3}_{\text{Think: } D_x(x \cdot 7) = 7} + \frac{1}{2x - 3y^2} \cdot D_x(2x - 3y^2)$$

$$= \boxed{y^3 + \frac{2}{2x - 3y^2}}$$

(b) Find $f_y(x, y)$

$$f(x, y) = \underbrace{xy^3}_\# + \ln(\underbrace{2x - 3y^2}_\#)$$

$$f_y(x, y) = \underbrace{x(3y^2)}_\# + \frac{1}{2x - 3y^2} \cdot D_y(\underbrace{2x - 3y^2}_\#)$$

7-factor doesn't disappear

$$\text{Think: } D_y(7y^3) = 7(3y^2)$$

$$= 3xy^2 + \frac{1}{2x - 3y^2} \cdot (-6y)$$

$$= \boxed{3xy^2 - \frac{6y}{2x - 3y^2}}$$

$$\text{Deal with later: } D(e^A) = e^A D(A)$$

Ex $f(x, y, z) = ye^{\overbrace{xy+yz}^\#}$. Find f_x and $f_x(0, 3, 4)$. $\overset{w}{x} \overset{w}{y} \overset{w}{z}$

$$f_x(x, y, z) = ye^{\overbrace{xy+yz}^\#} \cdot D_x(\underbrace{xy + \underbrace{yz}_\#}_\#)$$

$$= y + 0$$

$$= y$$

$$= \boxed{y^2 e^{xy+yz}}$$

$$f_x(0, 3, 4) = (3)^2 e^{(0)(3) + (3)(4)}$$

$$= \boxed{9e^{12}}$$

D) 2nd-Order PDs

$$\left. \begin{aligned} f_{xx} &= (f_x)_x = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} \right) = \frac{\partial^2 f}{\partial x^2} \text{ or } \underbrace{\frac{\partial^2 f}{\partial x^2}}_{\text{operator}} \\ f_{yy} &= (f_y)_y = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial y} \right) = \frac{\partial^2 f}{\partial y^2} \text{ or } \underbrace{\frac{\partial^2 f}{\partial y^2}}_{\text{operator}} \end{aligned} \right\} \Rightarrow \begin{array}{l} \text{Concavity} \\ \text{in } x, y \\ \text{directions} \end{array}$$

Mixed Partials

Larson: Start w/ variable nearest f

$$\left. \begin{aligned} f_{xy} &= (f_x)_y = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right) = \frac{\partial^2 f}{\partial y \partial x} \\ f_{yx} &= (f_y)_x = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \right) = \frac{\partial^2 f}{\partial x \partial y} \end{aligned} \right\} \begin{array}{l} \text{These are} \\ \text{equal where} \\ \text{both are} \\ \text{continuous.} \end{array}$$

These extend naturally to higher-order PDs.

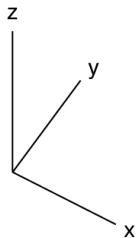
Ex

$$\begin{aligned} f(x,y) &= (3x+y^2)^5 \\ \textcircled{f}_x &: 5(3x+y^2)^4(3) & \textcircled{f}_y &: 5(3x+y^2)^4(2y) \\ &= 15(3x+y^2)^4 & &= 10y(3x+y^2)^4 \\ \textcircled{f}_{xx} &: 60(3x+y^2)^3(3) & \textcircled{f}_{xy} &: 60(3x+y^2)^3(2y) & \textcircled{f}_{yy} &: 10y \cdot 4(3x+y^2)^3(3) \\ &= 180(3x+y^2)^3 & &= 120y(3x+y^2)^3 & & \textcircled{f}_{yy} \\ &&&\text{Product Rule}&& \\ \textcircled{f}_{yy} &: [D_y(10y)] \cdot (3x+y^2)^4 + (10y) \cdot D_y[(3x+y^2)^4] &&&& \\ &= 10(3x+y^2)^4 + (10y) \cdot 4(3x+y^2)^3(2y) &&&& \\ &= 10(3x+y^2)^4 + 80y^2(3x+y^2)^3 &&&& \end{aligned}$$

16.3: PARTIAL DERIVATIVES

(Mathematica was used to produce these computer graphics.)

Note: The coordinate axes are oriented in an unusual way:



The large black dot indicates the point $(1, 2, 3)$.

Figure #1: Graph of $f(x,y) = -5x^2 + y^3$

Figure #2: The tangent plane to the graph of f at $(1, 2, 3)$

Figure #3: Combining #1 and #2

Figure #1

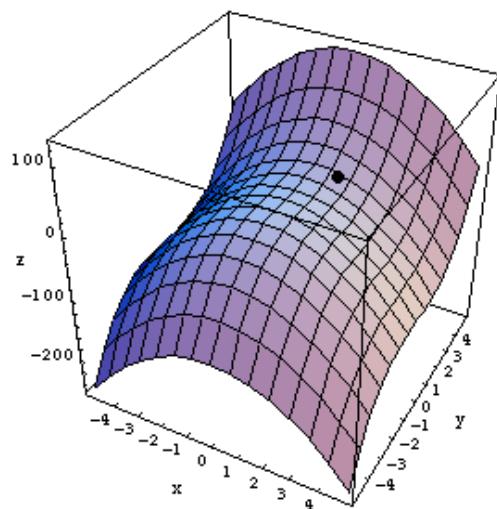


Figure #2

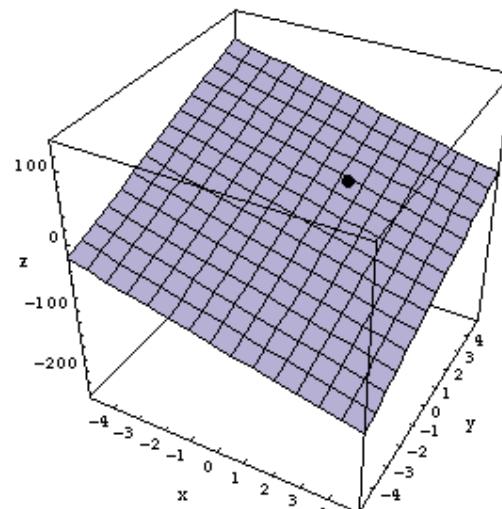
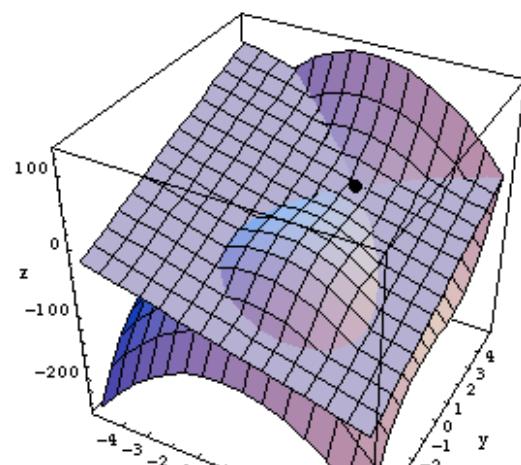


Figure #3



Find $\frac{\partial}{\partial y}$ of the following
(remember, x acts like a
constant):

- 1) $e^{xx}y^2$
- 2) $y^2e^{x^2y^3}$
- 3) $\frac{y^2}{\ln y}$
- 4) $\cos^3(xy^2)$
- 5) $\tan^{-1}(xy^2)$
- 6) $\sin^3 y$
- 7) $\sin(y^3)$
- 8) $\ln\sqrt{x^2+y^2}$

Answers

(not necessarily simplified):

$$1) \frac{\partial}{\partial y} \left(e^{xx} y^2 \right) = e^{xx} (2y)$$

e^{xx} acts like a constant multiplier for y^2

$$\begin{aligned} 2) \frac{\partial}{\partial y} \left(y^2 e^{x^2 y^3} \right) &= (2y) e^{x^2 y^3} + y^2 \left(e^{x^2 y^3} \bullet \frac{\partial}{\partial y} (x^2 y^3) \right) \\ &= (2y) e^{x^2 y^3} + y^2 \left(e^{x^2 y^3} \bullet x^2 (3y^2) \right) \end{aligned}$$

Product rule and chain rule

$$3) \frac{\partial}{\partial y} \left(\frac{y^2}{\ln y} \right) = \frac{(\ln y)(2y) - \left(y^2 \right) \left(\frac{1}{y} \right)}{\left(\ln y \right)^2}$$

Quotient rule: $\frac{\text{Lo} \bullet \text{D(Hi)} - \text{Hi} \bullet \text{D(Lo)}}{\text{Square of below}}$

As in 2), write $\frac{\partial}{\partial y}$ if you need the chain rule.

$$\begin{aligned} 4) \frac{\partial}{\partial y} \left[\cos^3(xy^2) \right] &= \frac{\partial}{\partial y} \left[\left(\cos(xy^2) \right)^3 \right] \quad (\text{Clearer notation}) \\ &= 3 \left[\cos(xy^2) \right]^2 \bullet \frac{\partial}{\partial y} \left[\cos(xy^2) \right] \\ &= 3 \left[\cos(xy^2) \right]^2 \bullet \left(-\sin(xy^2) \bullet \frac{\partial}{\partial y} (xy^2) \right) \\ &= 3 \left[\cos(xy^2) \right]^2 \bullet \left(-\sin(xy^2) \bullet x(2y) \right) \end{aligned}$$

Trig powers, power rule, chain rule (twice!)

$$5) \frac{\partial}{\partial y} \left(\tan^{-1}(xy^2) \right) = \frac{1}{1+(xy^2)^2} \bullet \frac{\partial}{\partial y} (xy^2)$$

$$= \frac{1}{1+(xy^2)^2} \bullet x(2y)$$

Rule: $\frac{\partial}{\partial y} \tan^{-1}(blah) = \frac{1}{1+blah^2} \bullet \frac{\partial}{\partial y}(blah)$

$$6) \frac{\partial}{\partial y} (\sin^3 y) = (\sin y)^3 \text{ (Clearer notation)}$$

$$= 3(\sin y)^2 \bullet \frac{\partial}{\partial y} (\sin y)$$

$$= 3(\sin y)^2 \bullet \cos y$$

$$7) \frac{\partial}{\partial y} (\sin(y^3)) = \cos(y^3) \bullet \frac{\partial}{\partial y} (y^3)$$

$$= \cos(y^3) \bullet 3y^2$$

As opposed to 6), we don't need the power rule here until the end.

$$\begin{aligned}
8) \quad & \frac{\partial}{\partial y} \left(\ln \sqrt{x^2 + y^2} \right) = \frac{1}{\sqrt{x^2 + y^2}} \bullet \frac{\partial}{\partial y} \left(\sqrt{x^2 + y^2} \right) \\
&= \frac{1}{\sqrt{x^2 + y^2}} \bullet \frac{\partial}{\partial y} \left[(x^2 + y^2)^{1/2} \right] \\
&= \frac{1}{\sqrt{x^2 + y^2}} \bullet \frac{1}{2} (x^2 + y^2)^{-1/2} \bullet \frac{\partial}{\partial y} (x^2 + y^2) \\
&= \frac{1}{\sqrt{x^2 + y^2}} \bullet \frac{1}{2} (x^2 + y^2)^{-1/2} \bullet 2y
\end{aligned}$$

Rule: $\frac{\partial}{\partial y} \ln(blah) = \frac{1}{blah} \bullet \frac{\partial}{\partial y}(blah)$

Roots as powers, power rule, chain rule

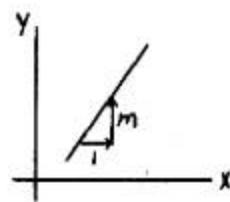
16.4: INCREMENTS and DIFFERENTIALS

Not in How to Ace

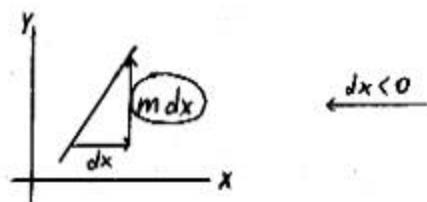
$$\Delta x, \Delta y, \Delta z, \Delta w \quad \quad dx, dy, dz, dw$$

(A) Interpreting Slope

$$f(x) = mx + b$$



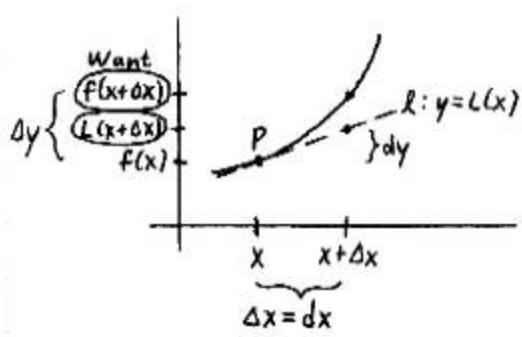
$$\begin{matrix} \text{run } 1 \\ \Rightarrow \text{rise } m \end{matrix}$$



$$\begin{matrix} \text{run } dx \\ \Rightarrow \text{rise } m dx \end{matrix}$$

$$\text{slope} = \frac{\text{rise}}{\text{run}} \Rightarrow \text{rise} = (\text{slope})(\text{run})$$

$$= f'(x) dx$$

(B) Review Calc I: $y = f(x)$ 

Find $L(x + \Delta x)$, a linear approximation for $f(x + \Delta x)$ based on a "seed" point $P(x, f(x))$ and its tangent line, l .

$$\text{Approx. } f(x + \Delta x) = f(x) + \Delta y$$

by $L(x + \Delta x) = f(x) + \underbrace{\Delta y}_{dy}$

$$\begin{aligned} \text{where } dy &= \text{rise along } l \\ &\text{as } x \rightarrow x + \Delta x \\ &= f'(x) dx \end{aligned}$$

⑥ Now, Calc III: $z = f(x, y)$

Based on a "seed" point $P(x, y, f(x, y))$ and
 its tangent plane to the surface (graph of f),
 best linear model
 around P

$$\begin{array}{c} z = L(x, y) \\ z = f(x, y) \end{array}$$

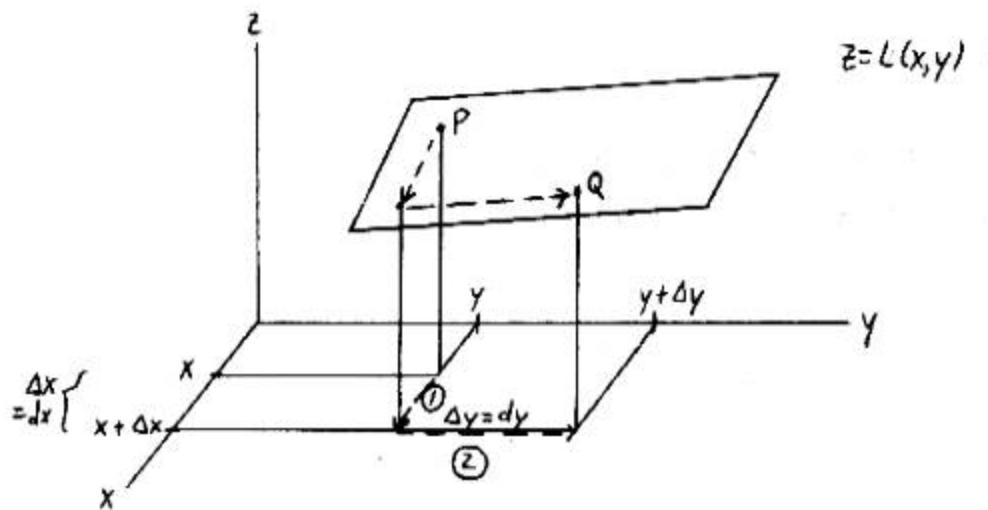
approx. $f(x + \Delta x, y + \Delta y) = f(x, y) + \Delta z$

actual change in z
 along surface as
 "shadow" $(x, y) \rightarrow$
 $(x + \Delta x, y + \Delta y)$

by $L(x + \Delta x, y + \Delta y) = f(x, y) + dz$

change in z
 along tangent plane

⑦ What is dz ?



$dz = \text{total differential of } z$

= change in z from P to Q

= (change in z from Stage ①) +
(Stage ②)

= (slope of tangent plane in x -direction)(run in x) +
(slope of tangent plane in y -direction)(run in y)

$$= f_x(x, y) dx + f_y(x, y) dy$$

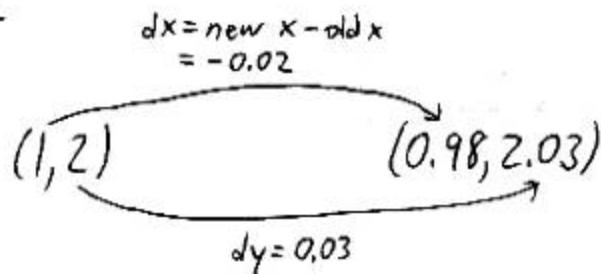
If $w = f(x, y, z)$,

$$dw = f_x dx + f_y dy + f_z dz$$

E) Ex

$$f(x,y) = x^2 + 3xy^2$$

Use the fact that $f(1,2) = 13$ to find a linear approx. for $f(0.98, 2.03)$.

Sol'n

$$dz = [f_x(1,2)]dx + [f_y(1,2)]dy$$

$$\begin{aligned} f_x(x,y) &= 2x + 3y^2 \\ f_x(1,2) &= 2(1) + 3(2)^2 \\ &= 14 \end{aligned}$$

$$\begin{aligned} f_y(x,y) &= 3x(2y) \\ &= 6xy \\ f_y(1,2) &= 6(1)(2) \\ &= 12 \end{aligned}$$

Once you do this work, you can quickly approx. $f(x,y)$ where $x \approx 1$ and $y \approx 2$. Multiple f evaluations?

$$\begin{aligned} &= (14)(-0.02) + (12)(0.03) \\ &= 0.08 \end{aligned}$$

$$\begin{aligned} L(0.98, 2.03) &= f(1,2) + dz \\ &= 13 + 0.08 \\ &= 13.08 \end{aligned}$$

Exact: 13.075846
 $\Delta z = 0.075846$

(F) Applications

- Given: Limited f info in a table
 ⇒ Estimate f_x, f_y at a "seed" in the table
 ⇒ Perform linear interpolations for f using differentials
 (near the seed)

Table:

x \ y	0	100	200
0	36	38	42
10	40	43	47
20	45	48	51

(G) Theory

Advanced Note

$$\left. \begin{array}{l} \text{f is differentiable at } (a,b) \text{ if we can write} \\ \Delta z = f_x(a,b)\Delta x + f_y(a,b)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y \\ \uparrow \quad \uparrow \\ \text{functions } \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \rightarrow 0 \\ \text{as } \Delta x \rightarrow 0, \Delta y \rightarrow 0 \end{array} \right\}$$

Stewart:
 f_x, f_y cont. but f, f_x, f_y
 not cont. at $(0,0)$

If f_x, f_y cont. on an open region $R \Rightarrow f$ is differentiable on R .
 (diff're)

↓
 ⇒ Linear approxs. tend to get better as $\Delta x \rightarrow 0$ and $\Delta y \rightarrow 0$.

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

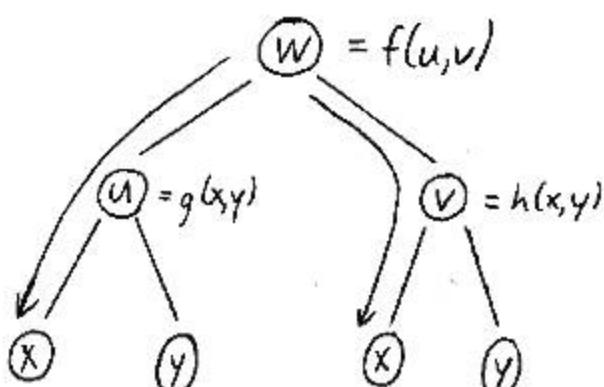
If f is diff're at $(a,b) \Rightarrow f$ is cont. there.

16.5: CHAIN RULESA) Intro

Assume funcs. are diff're where we care.

Calc I

$$\begin{array}{c} w = f(u) \\ \frac{dw}{du} | \\ u = g(x) \\ \frac{du}{dx} | \\ x \end{array} \quad \frac{dw}{dx} = \frac{dw}{du} \frac{du}{dx}$$

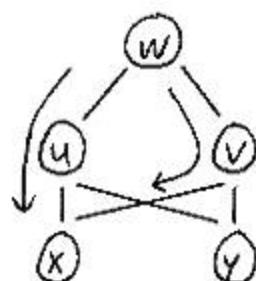
Calc III

Plinko model:

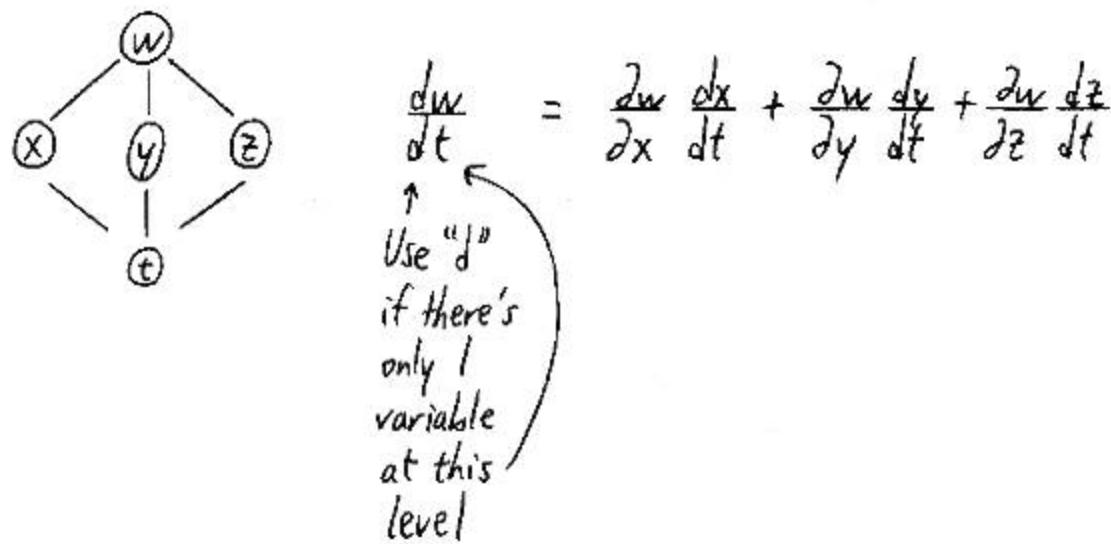
For $\frac{\partial w}{\partial x}$,
take products along
paths from w to x ,
and add them.
(due to 16.4 ideas)

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial w}{\partial v} \frac{\partial v}{\partial x}$$

or

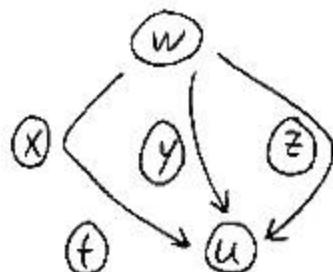


y y y



Ex Find $\frac{\partial w}{\partial u}$ if $w = 3x^2 + e^{4y} - x \ln z$, where $x = \sin(tu)$, $y = t^3 + u$, and $z = u^4$.

Sol'n



$$\begin{aligned}
 \frac{\partial w}{\partial u} &= \frac{\partial w}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial u} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial u} \\
 &= D_x (3x^2 + e^{4y} - x \ln z) \cdot D_u [\sin(tu)] \\
 &\quad + D_y (') \cdot D_u [t^3 + u] \\
 &\quad + D_z (') \cdot D_u [u^4] \\
 &= (6x - \ln z) \cdot [t \cos(tu)] \\
 &\quad + (4e^{4y}) \cdot (1) \\
 &\quad + (-x \cdot \frac{1}{z}) \cdot (4u^3)
 \end{aligned}$$

$$= (6x - \ln z) t \cos(tu) + 4e^{4y} - \frac{4xu^3}{z}$$

Sub into x, y, z .

$$= [6 \sin(tu) - \overline{\ln(u^4)}] t \cos(tu) \\ + 4e^{4(t^3+u)} \\ - \frac{[4 \sin(tu)] u^2}{u^4}$$

$$= [6 \sin(tu) - 4 \ln|u|] t \cos(tu) + 4e^{4(t^3+u)} \\ - \frac{4 \sin(tu)}{u}$$

Up to 13

In many problems,
easier than if we had subbed into x, y, z immediately!

(B) Getting Derivatives from Implicit Functions

(Shortcuts to 3.7 Method)

If $F(x, y) = 0$ determines a diff'e func. f
such that $y = f(x)$, then

$$\frac{dy}{dx} = -\frac{F_x}{F_y}$$

Think "negative reciprocal."
Proof based on Chain Rule.

If $F(x, y, z) = 0$ such that $z = f(x, y)$, then

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}, \quad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$$

Ex Find $\frac{\partial z}{\partial x}$ if $\underbrace{x^2 z + \tan(yz)}_{= F(x, y, z)} = 0$ (\leftarrow can't solve for z)

Sol'n

$$\frac{\partial z}{\partial x} = - \frac{F_x}{F_z}$$

$$= - \frac{2xz}{x^2 + y \sec^2(yz)}$$

Section 3.7 Method (for comparison :)

$$z = f(x, y)$$

$$\underbrace{D_x(x^2 z)}_{\text{Use Product Rule!}} + D_x[\tan(yz)] = 0$$

$$2xz + x^2 \frac{\partial z}{\partial x} + [\sec^2(yz)][y \frac{\partial z}{\partial x}] = 0$$

$$\therefore = -2xz$$

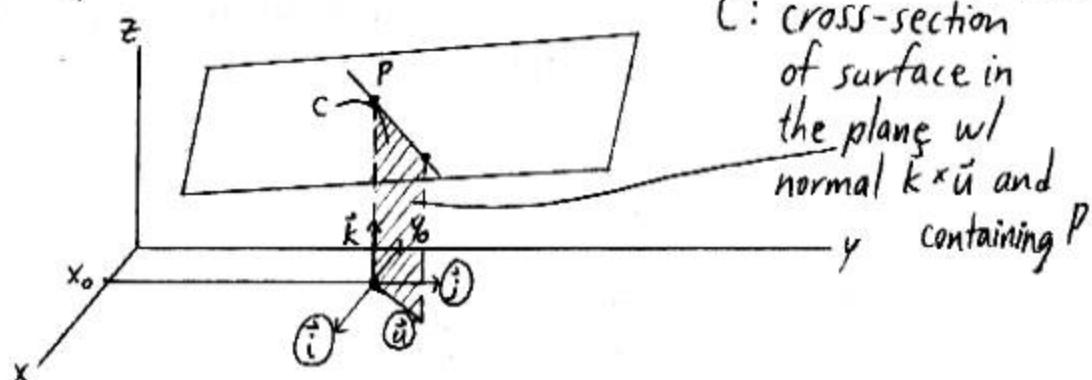
$$\frac{\partial z}{\partial x} [x^2 + y \sec^2(yz)] = -2xz$$

$$\frac{\partial z}{\partial x} = - \frac{2xz}{x^2 + y \sec^2(yz)} \quad \uparrow (\text{same})$$

16.6: DIRECTIONAL DERIVATIVES (DDs)

(A) Intro

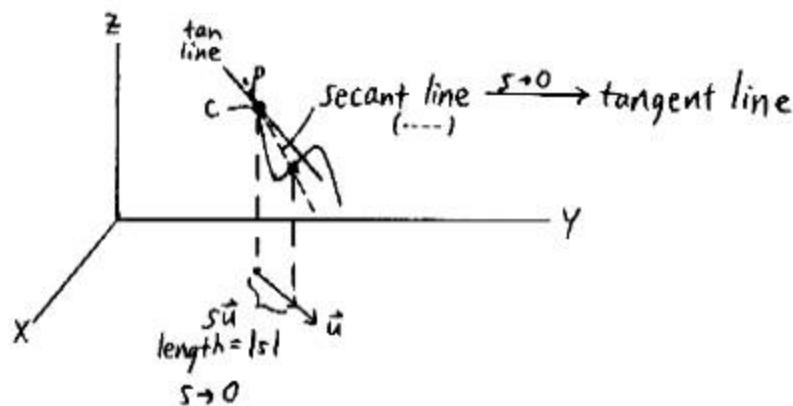
Consider the tangent plane to the graph of $z = f(x, y)$ at P ,



$f_x(x_0, y_0) =$ DD of f at P in the direction of \vec{i}
 $f_y(x_0, y_0) =$

$D_{\vec{u}} f(x_0, y_0) =$
 $=$ slope along tangent plane at
 P in the direction of $\vec{u} = \langle u_1, u_2 \rangle$,
 a unit vector indicating
 "compass direction"

$$\text{Def'n} \quad = \lim_{s \rightarrow 0} \frac{f(x_0 + su_1, y_0 + su_2) - f(x_0, y_0)}{s}$$



Not weighted
average of
 f_x, f_y , since
 $u_i + u_j \neq 1$
($=1$ if $i=j$)

def, like 2

$$\begin{aligned}
 D_{\vec{u}} f(x, y) &= f_x(x, y) u_1 + f_y(x, y) u_2 \\
 &= \langle f_x(x, y), f_y(x, y) \rangle \cdot \langle u_1, u_2 \rangle \\
 &= \vec{\nabla} f(x, y) \cdot \vec{u} \\
 &\quad \text{where } \vec{\nabla} f(x, y) = \text{the gradient of } f \\
 &= \langle f_x(x, y), f_y(x, y) \rangle
 \end{aligned}$$

Sketch of Proof of ④

$$\begin{array}{ccc}
 \textcircled{w} = f(h, v) & & \\
 \swarrow & & \searrow \\
 \textcircled{h} = x + su_1 & & \textcircled{v} = y + svu_2 \\
 & & \\
 & & \textcircled{s}
 \end{array}$$

$$\begin{aligned}
 \frac{dw}{ds} &= \underbrace{\frac{\partial w}{\partial h} \frac{dh}{ds}}_{=f_h = f_x \text{ at } s=0} + \underbrace{\frac{\partial w}{\partial v} \frac{dv}{ds}}_{=f_v = f_y \text{ at } s=0} \\
 &= f_x + f_y
 \end{aligned}$$

$$\text{Ex } f(x,y) = 2x^2 + y^2$$

a) Find $\vec{\nabla}f(2,3)$.

$$\vec{\nabla}f(x,y) = \langle f_x(x,y), f_y(x,y) \rangle$$

$$= \langle 4x, 2y \rangle$$

$$\vec{\nabla}f(2,3) = \langle 4(2), 2(3) \rangle$$

$$= \boxed{\langle 8, 6 \rangle}$$

b) Find the DD of f at $(2,3)$ in the direction of $\vec{a} = \langle -3, 1 \rangle$.

Find \vec{u} , the unit vector in the direction of \vec{a} .

$$\vec{u} = \frac{\vec{a}}{\|\vec{a}\|} = \frac{\langle -3, 1 \rangle}{\sqrt{(-3)^2 + (1)^2}}$$

$$= \frac{1}{\sqrt{10}} \langle -3, 1 \rangle$$

$$D_{\vec{u}}(2,3) = \vec{\nabla}f(2,3) \cdot \vec{u}$$

$$= \langle 8, 6 \rangle \cdot \frac{1}{\sqrt{10}} \langle -3, 1 \rangle$$

$$= \frac{1}{\sqrt{10}} (-24 + 6)$$

$$= -\frac{18}{\sqrt{10}}$$

$$\approx -5.7$$

③ Find the DD of f at $(2,3)$ in the direction of $\langle -3, 4 \rangle$

Normalize
↓
 \vec{u} (a new \vec{u})

$$\begin{aligned} D_{\vec{u}}(2,3) &= \vec{\nabla}f(2,3) \cdot \vec{u} \\ &= \underbrace{\langle 8, 6 \rangle \cdot \frac{\langle -3, 4 \rangle}{\|\langle -3, 4 \rangle\|}}_0 \\ &= 0 \end{aligned}$$

When do 2 vectors have a "u." of 0,
geom. speaking!

Note $\vec{\nabla}f(2,3) \perp \langle -3, 4 \rangle$
 $\text{DD}=?$ $\text{DD}=0$

③ Comparing DDs

The DD of f at (x,y) is maximized in the direction of $\vec{\nabla}f(x,y)$, (steepest) the direction of fastest increase of f . The corresponding DD = $\|\vec{\nabla}f(x,y)\|$.

minimized	$-\vec{\nabla}f(x,y)$	(steepest)
decrease	$-\ \vec{\nabla}f(x,y)\ $	

Why?

$$\cos(-\theta) = \cos \theta$$



$$D_{\vec{u}} f(x,y) = \vec{\nabla}f(x,y) \cdot \vec{u}$$

$$= \|\vec{\nabla}f(x,y)\| \|\vec{u}\| \cos \theta$$

$$\text{Max: } = 1 \quad (\theta=0) \quad \xrightarrow{\vec{u} \parallel \vec{\nabla}f}$$

$$\text{Min: } = -1 \quad (\theta=\pi) \quad \xleftarrow{\vec{u} \parallel -\vec{\nabla}f}$$

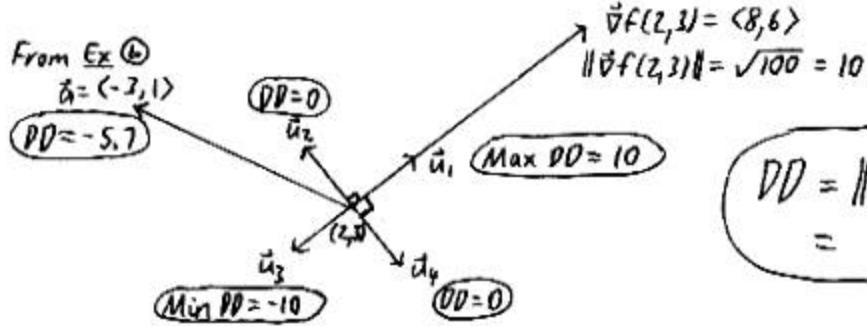
$$\text{Note: } = 0 \quad (\theta=\frac{\pi}{2})$$

$$\text{if } 0 \leq \theta \leq \pi \quad \xrightarrow{\vec{u} \perp \vec{\nabla}f}$$

Old Ex $f(x,y) = 2x^2 + y^2$

(Not to scale) \rightarrow

Are you surprised \rightarrow
it's this negative?



$$\begin{aligned} DD &= \|\vec{\nabla}f(x,y)\| \cos \theta \\ &= 10 \cos \theta \text{ here} \end{aligned}$$

Note The DD changes continuously but not steadily wrt θ .

fastest near $\theta=\frac{\pi}{2}$ (\vec{u}_2, \vec{u}_4)

slowest near $\theta=0, \pi$ (\vec{u}_1, \vec{u}_3)

Why? $D_\theta(DD) = -\|\vec{\nabla}f(x,y)\| \sin \theta$

≈ 0
most extreme

$\textcircled{DD=0}$

Level curve (LC) of $f(x,y) = 2x^2 + y^2$ through $(2,3)$:

Find k

$$\begin{aligned} k &= f(2,3) \\ &= 2(2)^2 + (3)^2 \\ &= 17 \end{aligned}$$

$$\frac{x^2}{\frac{17}{2}} + \frac{y^2}{17} = 1$$

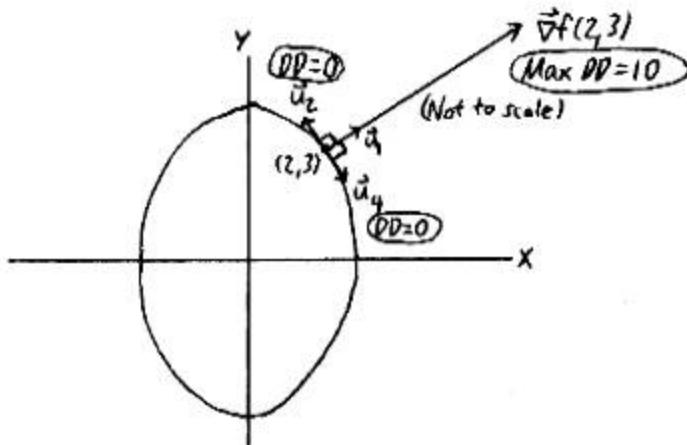
$$\downarrow \quad \downarrow$$

$$b \approx 2.9 \quad a \approx 4.1$$

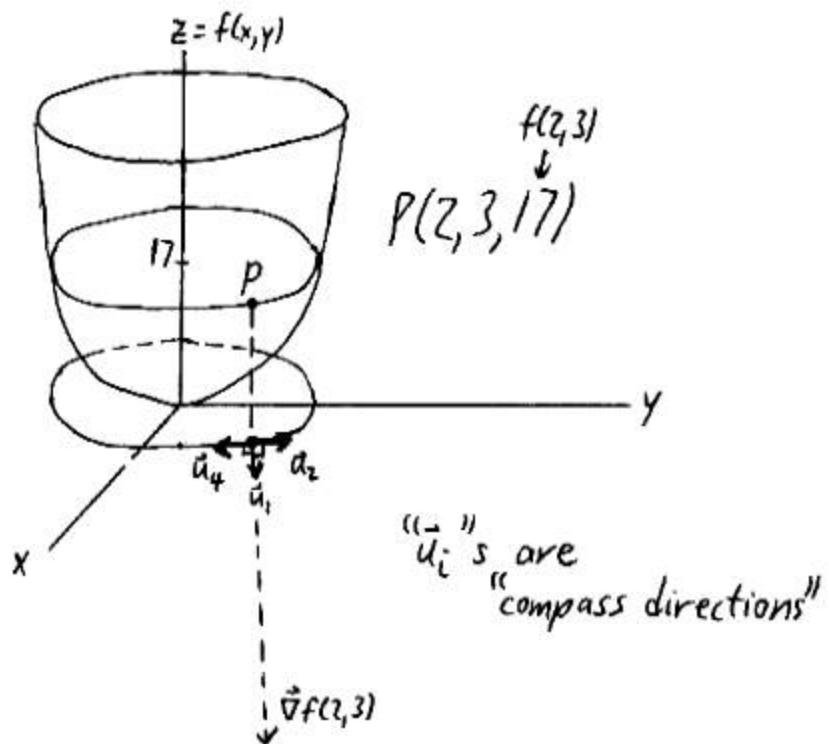
LC: $2x^2 + y^2 = 17$

$f(x,y) = 17$, a constant, for all (x,y) on LC.

At $(2,3)$, in
which directions
will $DD=0$?



$DD=0$ at $(2,3)$ in "tangent directions" to the LC through $(2,3)$.



Path of steepest ascent along the surface:

strategy?

Keep going in the direction of $\vec{\nabla}f(x, y)$ on your compass.
 may change
 as you move

Hard fall

'descent'
 $-\vec{\nabla}f(x, y)$

(C) $w = f(x, y, z)$

$$\vec{\nabla}f = \langle f_x, f_y, f_z \rangle$$

$$D_{\vec{u}} f = \vec{\nabla}f \cdot \vec{u}$$

unit $\langle u_1, u_2, u_3 \rangle$

Level curves \rightarrow Level surfaces (16.7)

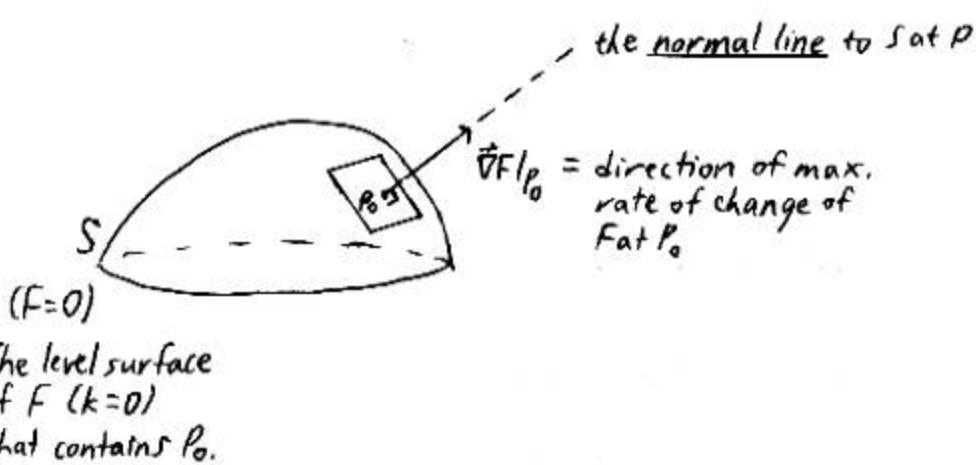
16.7: TANGENT PLANES and NORMAL LINES

Let S be the graph of $F(x, y, z) = 0$.
 Let $P_0(x_0, y_0, z_0)$ be a point on S .

If $\vec{F} = \langle F_x, F_y, F_z \rangle$ is cont.,

then $\vec{F}|_{P_0}$ \perp (the tangent plane to S at P_0),
 at

#27: Every
normal line
to a sphere
passes through
the center.



What's an eq. for the tangent plane at P_0 ?

Ingredients

A point: $P_0(x_0, y_0, z_0)$

A normal: $\vec{F}|_{P_0} = \langle F_x|_{P_0}, F_y|_{P_0}, F_z|_{P_0} \rangle \leftarrow \vec{n}$

From 14.5,

$$(F_x|_{P_0})(x - x_0) + (F_y|_{P_0})(y - y_0) + (F_z|_{P_0})(z - z_0) = 0$$

- Ex ① Find an eq. for the tangent plane to the graph of $z = 2x^2 + y^2$ at $P_0(2, 3, 17)$. ← from 16.6
 ② Find eqs. for the normal line at P_0 .

Sol'n

$$z = 2x^2 + y^2$$

Isolate 0 on one side.

$$\begin{aligned} 0 &= \underbrace{2x^2 + y^2 - z}_{= F(x, y, z)} \\ &= F(x, y, z) \end{aligned}$$

$$\vec{\nabla}F = \langle 4x, 2y, -1 \rangle$$

$$\begin{aligned} \vec{\nabla}F|_{P_0} &= \langle 4(2), 2(3), -1 \rangle \\ &= \langle 8, 6, -1 \rangle \quad \leftarrow \text{"normal vector"} \end{aligned}$$

Projection in
xy-plane
is $8\hat{i} + 6\hat{j}$
= $\vec{\nabla}f(2, 3)$,
where
 $f(x) = 2x^2 + y^2$.

① Tangent plane

$$8(x-2) + 6(y-3) - (z-17) = 0$$

② Normal line

$$\boxed{\begin{cases} x = 2 + 8t \\ y = 3 + 6t \\ z = 17 - t \end{cases}, t \in \mathbb{R}}$$

$\uparrow \quad \uparrow$
 $P_0 \quad \vec{n}$

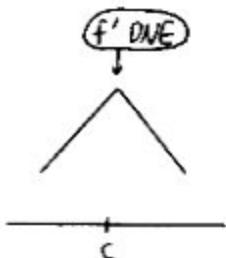
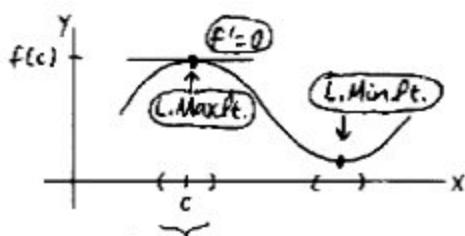
16.8: OPTIMIZATION I

(A) Local/Relative Extrema of $f(x, y)$

L.

Max, Min

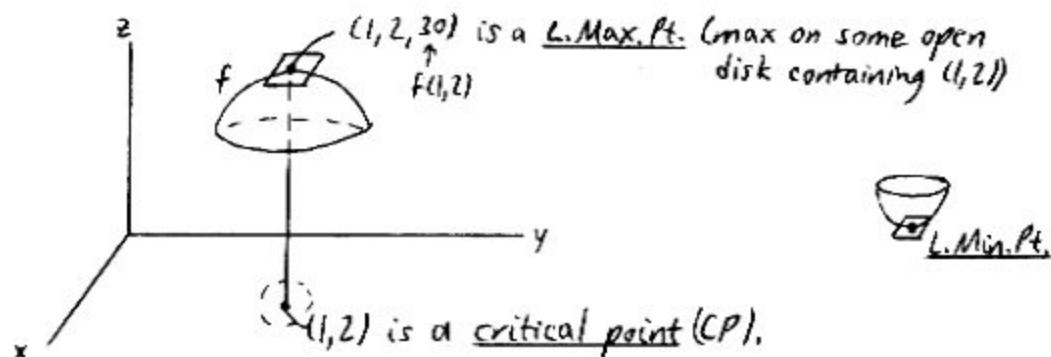
Calc I: $y = f(x)$



$f(c)$ is the max value of f on some open interval containing c , a critical #

A critical # is a # in $\text{Dom}(f)$ where $f' = 0$ or DNE.
These are the only #s (candidates) where L. Max./Min. may occur. (Not "must": ↗)

Now: $z = f(x, y)$



$(1, 2)$ is a critical point (CP).

f has a L. Max. of 30 at $(1, 2)$.

not closed
(we exclude
boundary pts.)

Note 1: There is a horizontal tangent plane at $(1, 2, 30)$.

$$\Rightarrow \begin{array}{c} \partial f = f_y = 0 \\ \partial f = f_x = 0 \end{array} \quad \left(\begin{array}{l} \text{If } \vec{v} \cdot \vec{\nabla} f = 0 \text{ for all } \vec{v}, \\ \text{then } f \text{ is diff're at } P \end{array} \right) \quad \left. \begin{array}{l} \text{All } \partial f \text{ at } P \text{ are } 0. \\ f \text{ is diff're at } P \end{array} \right\}$$

Note 2:

Why does this make sense?

$$\vec{\nabla} f = \langle f_x, f_y \rangle$$

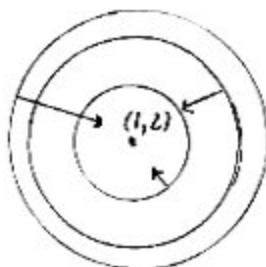
$$\vec{\nabla} f(1, 2) = \langle 0, 0 \rangle = \vec{0}$$

If you're at the North Pole, can you go further North?

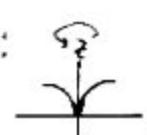
If $\vec{\nabla} f \neq \vec{0}$, then f can increase in that direction, and decrease in the opposite direction.

We can't be at an L. Max./Min. Pt.

Level Curves of f



$\vec{\nabla} f$ shrinks as we move towards $(1, 2)$.

Note 3: $f(x, y) = \sqrt[3]{x^2 + y^2}$ from:  $z = \sqrt[3]{x^2 + y^2}$

L. Min. Pt.
 f_x, f_y DNE

(In fact, all ∂f s at $(0, 0)$ DNE.)

Def'n (a, b) is a CP \iff

① (a, b) is in $\text{Dom}(f)$

② $f_x(a, b) = 0$ and $f_y(a, b) = 0$ (i.e., $\vec{\nabla} f(a, b) = \vec{0}$)

or ③ either DNE

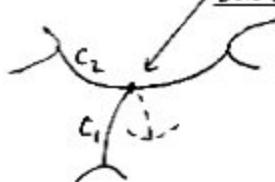
(i.e., $\vec{\nabla} f(a, b)$ DNE)

CPs are the only places where L, Max./Min. Pts. can occur.

Ex 2 (p. 863) CP where neither occurs

$$f(x, y) = y^2 - x^2 \quad (\text{Hyp. paraboloid})$$

saddle point (SP): Max on C_1 , Min on C_2



Discriminants
help us
classify.
 $b^2 - 4ac$ helped
us classify
roots of a
quadratic func.
as real or
imaginary.

⑧ Classifying CPs

Assume the 2nd PDs of f are cont. where we care
 $\Rightarrow f_{xy} = f_{yx}$

The discriminant of $f = "D"$ or " $D(x, y)$ "

$$= \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}$$

$$= f_{xx} f_{yy} - (f_{xy})^2$$

real,
sym. matrix
all evals
real

Do we care if
we switch
 f_{xy}, f_{yx} ?

Larson 8th
(ed) - can
find C₂₂
in higher dims,
but test is
ugly

2nd Derivative Test for $f(x,y)$

At a CP (a,b) where $\vec{\nabla} f = \vec{0}$, not DNE,

*① If $D > 0$,

①a if $f_{xx} < 0 \Rightarrow$ concave down \frown (word association works)
 \Rightarrow L. Max. at (a,b) fails

①b $>$ up \smile
L. Min.

**② If $D < 0 \Rightarrow$ saddle pt. (SP)

③ If $D = 0 \Rightarrow$ no info

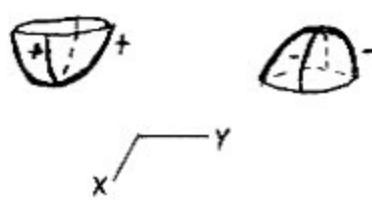
In Calc I, SOT,
what did
 $f''=0$ tell us?

* For ①, you can use f_{yy} .

If $f_{xx}, f_{yy} > 0$
and f_{xy}
not too
influential
 $\Rightarrow D > 0$

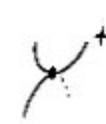
If $D = \underbrace{f_{xx} f_{yy}}_{>0} - (f_{xy})^2 > 0$, then

$\Rightarrow f_{xx}, f_{yy}$ have same sign



x, y -concavities
both up or both down
and f_{xy} not too
influential

**② Ex $f_{xx} < 0, f_{yy} > 0 \Rightarrow D < 0 \Rightarrow$ SP



① Exs

Ex Find the local extrema and saddle points of
 $f(x,y) = -x^2 - y^3 - 6x + 3y + 4$.

M121 #18
3rd ed.

Step 1: Find CPs.

Technically,
 $f_x(x,y)$

$$\left. \begin{array}{l} f_x = \underbrace{-2x - 6}_{\text{never DNE}} \stackrel{\text{set}}{=} 0 \\ f_y = \underbrace{-3y^2 + 3}_{\text{never DNE}} \stackrel{\text{set}}{=} 0 \end{array} \right\} \text{Solve system.}$$

$$\begin{aligned} -2x - 6 &= 0 & \text{and} & \quad -3y^2 + 3 = 0 \\ x &= -3 & & \quad y^2 = 1 \\ & & & \quad y = \pm 1 \end{aligned}$$

$$(CP_s : (-3, 1) \\ (-3, -1))$$

Note: If we had...

$$\left\{ \begin{array}{l} x+y=0 \Leftrightarrow y=-x \\ x-y^2=0 \end{array} \right.$$

$$\Leftrightarrow \left\{ \begin{array}{l} y=-x \\ x-x^2=0 \\ x(1-x)=0 \\ x=0 \Rightarrow y=0 \\ x=1 \Rightarrow y=-1 \end{array} \right.$$

$$(P_s : (0,0) \\ (1,-1))$$

Step 2: Find f_{xx}, D .

$$\begin{array}{l} f_x = -2x - 6 \\ f_{xx} = \textcircled{-2} \end{array}$$

$$f_{xy} = f_{yx} = 0$$

$$\begin{array}{l} f_y = -3y^2 + 3 \\ f_{yy} = -6y \end{array}$$

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} -2 & 0 \\ 0 & -6y \end{vmatrix} = \textcircled{12y}$$

Step 3: Classify CPs.

CP	$D = 12y$	$f_{xx} = -2$	Conclusion
$(-3, 1)$	$12(1) = 12 > 0$	$-2 \text{ } \textcircled{\times}$	L. Max.
$(-3, -1)$	$12(-1) = -12 < 0$	(irrelevant)	SP

Step 4: Find f values at CPs.

L. Max. Pt. at $(-3, 1, f(-3, 1))$ $(-3, 1, 15)$ <small>\leftarrow L. Max. Value</small>
SP at $(-3, -1, f(-3, -1))$ $(-3, -1, 11)$

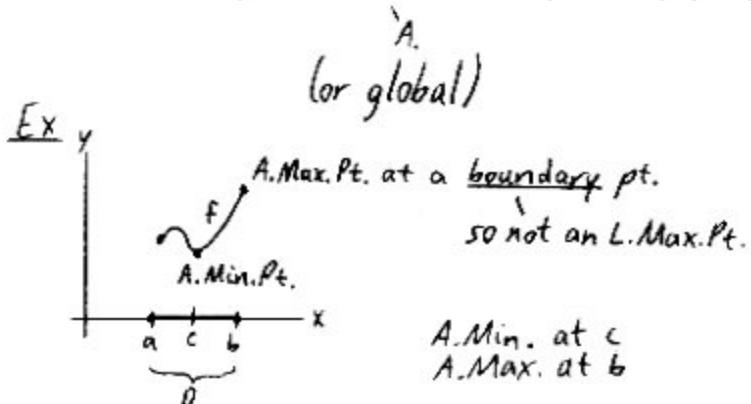
Larson #59
 $x^4 - 2x^2 + y^2$
has 2 rel. min.
but no rel. max.

① What if the domain, D , is closed? (Not on tests)

Calc I

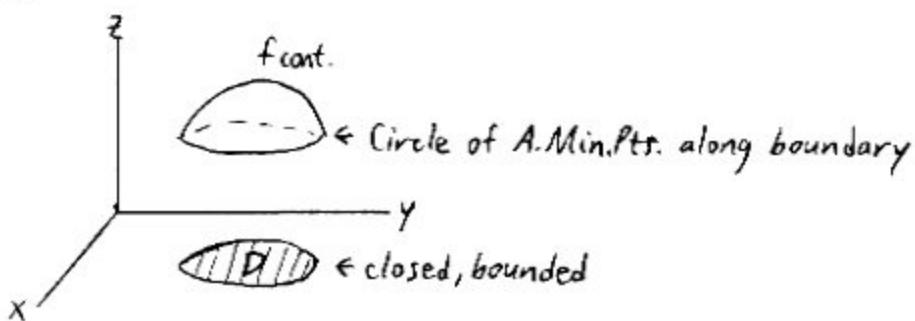
Extreme Value Thm. (EVT)

If f is cont. on $[a, b]$, a closed interval
 \Rightarrow There exist absolute max. and min. in $[a, b]$.



Now

Unbounded
Bounded = is a subregion of some disk



If D is open \Rightarrow no boundary extrema to worry about

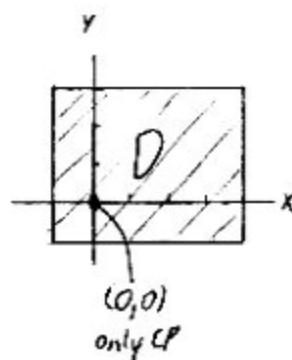
If D is closed \Rightarrow examine boundary for possible A. Min./Max. Pts.

EVT Extension: If f is cont. on a closed D
 \Rightarrow There exist A. Max. and A. Min. in D .

Ex (#25) Find absolute extrema of $f(x,y) = x^2 + 2xy + 3y^2$
on $D = \{(x,y) \mid -2 \leq x \leq 4 \text{ and } -1 \leq y \leq 3\}$

Collect candidates where A.Max./Min. Pts. might appear.

① Find CPs in D (excluding boundary).



② Examine the 4 sides.

closed
interval
→ redundant
work; see C

Ex $\boxed{D} \leftarrow \text{an open interval}$

⇒
Splitter
Param w/t

$$\begin{aligned} f(4,y) &= (4)^2 + 2(4)y + 3y^2 \\ &= \underbrace{16 + 8y}_{g(y)} + 3y^2 \end{aligned}$$

$$\text{Calc I: } g'(y) = 0 \Rightarrow y = -\frac{4}{3}$$

but $(4, -\frac{4}{3})$ is not in D , so toss it!

③ Find corners of D .

④ Compare the f values at all our candidates.

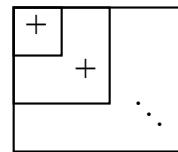
Highest f value \Rightarrow A.Max. value on D ; Lowest \Rightarrow A.Min.

PART E: FOOTNOTES

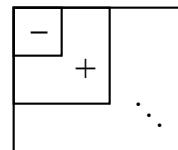
Extending the 2nd Derivative Test

If you have a nice function of n variables, you will construct an $n \times n$ real symmetric matrix consisting of n th-order partial derivatives; such a matrix only has real eigenvalues. When classifying a critical point (CP), we consider the signs of the determinants of all the upper left square submatrices (1×1 , 2×2 , etc.).

- If they are all positive, the matrix is called positive definite, and all of its eigenvalues are positive. The CP corresponds to a local min.



- If they alternate in sign from negative to positive, etc., the matrix is called negative definite, and all of its eigenvalues are negative. The CP corresponds to a local max.



- If they are all nonzero, and neither of the two above configurations occur, then the CP corresponds to a saddle point (SP).

Observe that the notes on 16.8.4 are consistent with all of this.

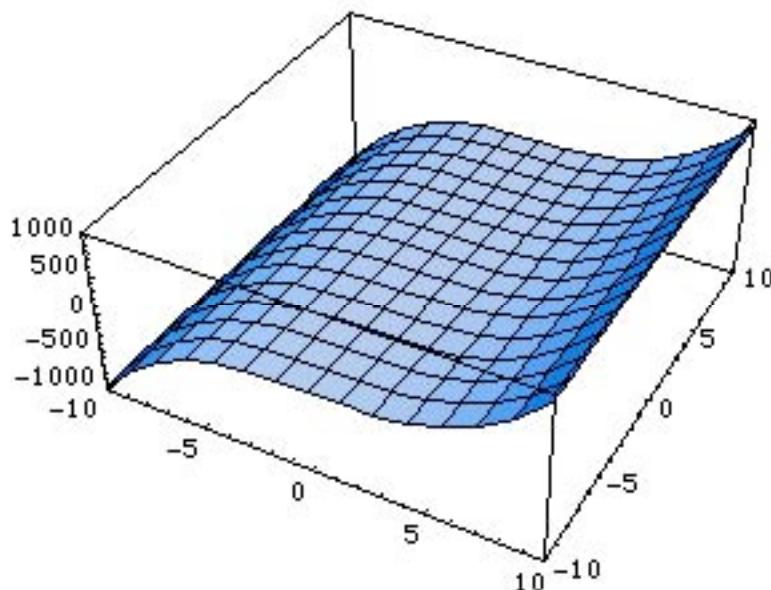
Defining a Saddle Point (SP)

The Harper Collins Dictionary of Mathematics:

“A point on a surface that is a maximum in one planar cross-section and a minimum in another.”

Visualizing a hyperbolic paraboloid helps.

The definition may vary. Are degenerate "ties" allowed along a cross-section, like for horizontal lines? Also, for example, are the points along the y -axis saddle points if we have the graph of the "snake cylinder" $f(x, y) = x^3$?



Orientation of axes: $\begin{smallmatrix} y \\ x \end{smallmatrix}$

That's debatable. Using the *Harper Collins* definition, I don't believe they would be; the thing just doesn't look like a "saddle" along the y -axis. But it is true that there are higher and lower points "immediately around" those points. Incidentally, $D = 0$ everywhere for this function, so the 2nd Derivative Test says nothing.

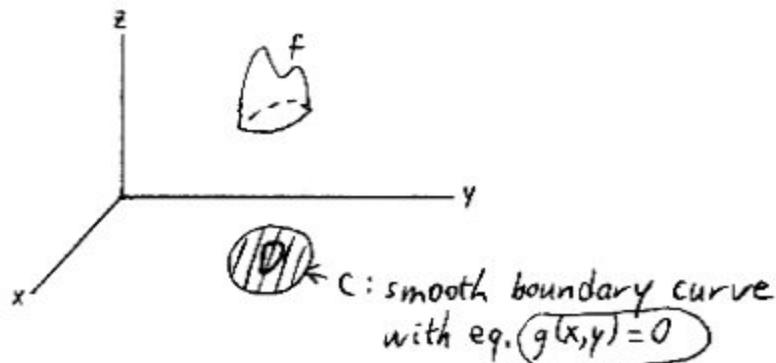
See: http://en.wikipedia.org/wiki/Saddle_point

16.9: CONSTRAINED OPTIMIZATION - LAGRANGE MULTIPLIERS

(A) Intro

In 16.8 (1)

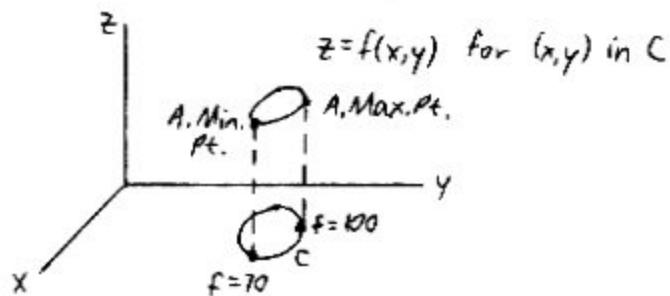
See Larson
6^{ed} p. 906



I'd like to analyze f along C .

Now, what if C is my [restricted] domain?

Like a stripe
along the
surface



"where we care":
In vicinity of
pts. along C ?
16.4:

f_x, f_y cont.
in open region
 $\exists f$ diff'rent
 $\Rightarrow f$ cont.

Another EVT Extension

If f is cont. on a closed curve (like δ)

or on a curve that includes its endpoints (like γ)

\Rightarrow There exist A.Max. and A.Min. along the curve.

Note: This extends to surfaces and their boundaries
in higher dims.

Assume ∇f is cont., $\nabla g \neq 0$ where we care.

Goal: Find L. or A. Max./Min. of $f(x,y)$ subject to the constraint $g(x,y)=0$. (Calc I: Pigpen problems!)

Note 1 If you can solve $g(x,y)=0$ for y in terms of x or x y

$$\begin{aligned} & \text{100 ft. of fencing } \square \text{ } y \\ & \text{Maximize Area } f(x,y) = xy \\ & \text{s.t. } 2x+2y=100 \\ & (\text{i.e., } 2x+2y-100=0) \\ & g(x,y) \\ & \Rightarrow y = 50-x \end{aligned}$$

If can solve
for y in
terms of x
 \Rightarrow let $t=x$, or
 $x=t$.

or If you can parameterize x and y in terms of t , then you can do it, sub into $f(x,y)$, and use Calc I.

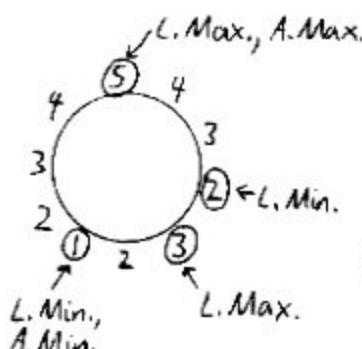
$$\begin{array}{c} x \quad y \\ \downarrow \quad \uparrow \\ t \end{array} \quad \begin{array}{c} y \\ \oplus x \\ x = \cos t \\ y = \sin t \end{array}$$

Note 2

Ex (Not the "f" from 16.9.1.)

Labels are f values:

To trace the corresp.
graph of f ,
move your
finger so
it has height
 z or f .



(OK, maybe the L.Max.
value is 3.1, not 3.
Shut up! :)

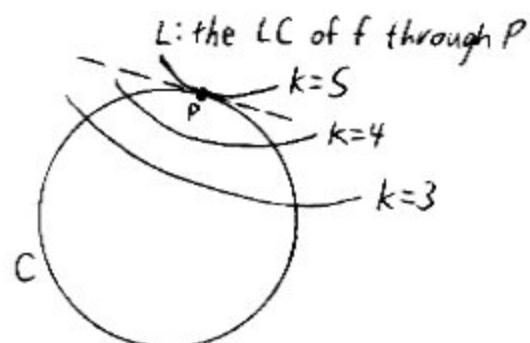
If C is a closed curve like \circlearrowleft or is like \circlearrowright , then A. Max./Min. are L. Max./Min.
(We're assuming f is cont. where we care.)

If $\overset{P}{\curvearrowleft} \overset{Q}{\curvearrowright} C$, then check for possible A. Max./Min.
at P, Q . Can't have L. Max./Min. there.

⑧ Lagrange's Thm.

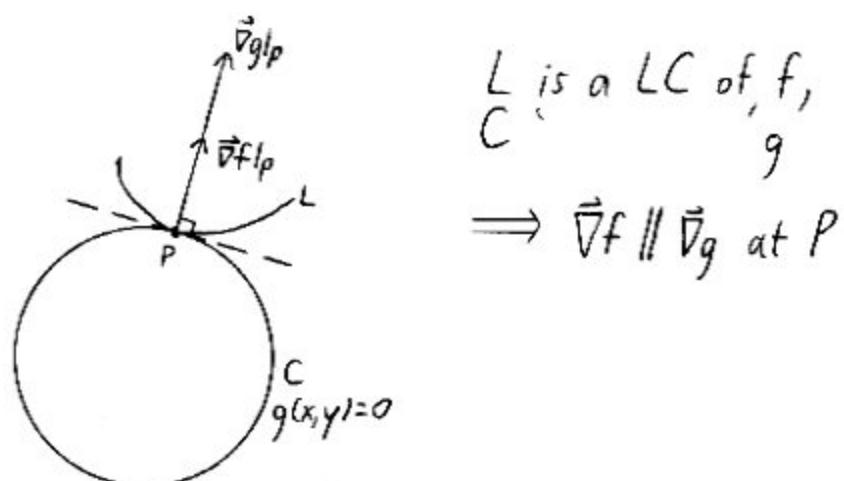
Idea Consider the level curves ($L(s)$) of f .

Where are the local extrema of f on C ?



Where the $L(s)$ of f barely touch C . (Why? See 16.9.11)

At P , L and C share the same tangent line. ---



Lagrange's Thm.

If P is a L.Max./Min. locale along C ,
then there is a real #, λ (lambda), such that

$$\vec{\nabla}f = \lambda \vec{\nabla}g \text{ at } P.$$

λ is a Lagrange multiplier.

Proof (Optional)

Let $\vec{r}(t) = \langle x(t), y(t) \rangle$ be a smooth param. of C .
 Let $h(t) = f(x(t), y(t))$.

$$\circlearrowleft_{t^{(x,y)}} h(t) = f(x,y)$$

$h'(t)$ or $\frac{dh}{dt} = 0$ at P , a L.Max./Min. locale along C .
 (can't be "DNE" where $\vec{\nabla}f$ cont.)

By Chain Rule, \vec{x}'_t^h

At P ,

$$\frac{dh}{dt} = \underbrace{\frac{dh}{dx} \frac{dx}{dt}}_{=f_x} + \underbrace{\frac{dh}{dy} \frac{dy}{dt}}_{=f_y} = 0$$

$$\underbrace{\langle f_x, f_y \rangle}_{\vec{\nabla}f} \cdot \underbrace{\langle \frac{dx}{dt}, \frac{dy}{dt} \rangle}_{\vec{r}'} = 0$$

$$\left. \begin{array}{l} \vec{r} \text{ is tangent to } C \text{ at } P \\ C \text{ is a LC of } g \end{array} \right\} \Rightarrow \left. \begin{array}{l} \vec{\nabla}f \perp \vec{r}' \\ \vec{\nabla}g \perp \vec{r}' \end{array} \right\} \Rightarrow \left. \begin{array}{l} \vec{\nabla}f \parallel \vec{\nabla}g \\ \vec{\nabla}f = \lambda \vec{\nabla}g \end{array} \right\}$$

for some real λ

(C) Method

Goal: Find L. or A. Max./Min. of $f(x,y)$
subject to $g(x,y)=0$.

To find the candidates (x,y) for L.Max./Min. locales,

$$\text{solve } \begin{cases} \vec{\nabla}f(x,y) = \lambda \vec{\nabla}g(x,y) \\ g(x,y) = 0 \end{cases} \quad \leftarrow \text{ensures } (x,y) \text{ is on } C$$

for (x,y, λ)

can differ for different candidates (x,y) ;
don't have to find λ (means to end)

If you're looking for A.Max./Min.,
examine any endpoints of C .

Method extends to higher dimensions. ($LC_s \rightarrow LS_s$)

Note: If there are 2 constraints, $g(x_1, \dots, x_n) = 0$
and $h(x_1, \dots, x_n) = 0$,
I'll omit

$$\text{solve } \begin{cases} \vec{\nabla}f = \lambda \vec{\nabla}g + \mu \vec{\nabla}h \\ g = 0 \\ h = 0 \end{cases} \quad \leftarrow \begin{array}{l} \text{linear combo of} \\ \vec{\nabla}g, \vec{\nabla}h \end{array}$$

for (x, y, z, λ, μ)

Intersection
of $g=0, h=0$

In Ex. 4 on
pp. 879-880,
can param.
this,
use Calc I.

Stewart
12.8, #3
Math 20C
not in ET, 5e

DEx Find the L. Max./Min. of $f(x, y) = xy$
subject to $9x^2 + y^2 = 4$.

Sol'n

Not really
necessary (here, but good form).
We then use " $g=0$ ".

$$\underbrace{9x^2 + y^2 - 4}_g = 0 \quad (\text{isolate } 0.)$$

$$\text{Solve } \begin{cases} \vec{\nabla}f(x, y) = \lambda \vec{\nabla}g(x, y) \\ g(x, y) = 0 \end{cases} \leftarrow \textcircled{*}$$

$$\vec{\nabla}f(x, y) = \lambda \vec{\nabla}g(x, y)$$

$$\langle f_x, f_y \rangle = \lambda \langle g_x, g_y \rangle$$

$$\langle y, x \rangle = \lambda \langle 18x, 2y \rangle$$

$$\text{Solve } \begin{cases} \textcircled{1} & y = \lambda(18x) \\ \textcircled{2} & x = \lambda(2y) \\ \textcircled{*} & 9x^2 + y^2 - 4 = 0 \end{cases}$$

$$\textcircled{1} \quad y = \lambda(18x) \Rightarrow \lambda = \frac{y}{18x} \quad (\text{if } x \neq 0) \quad \text{or} \quad x = 0$$

$\Downarrow y = \lambda(18x)$
 $y = 0$

$$\textcircled{2} \quad x = \lambda(2y) \Rightarrow \lambda = \frac{x}{2y} \quad (\text{if } y \neq 0) \quad \text{or} \quad y = 0$$

$\Downarrow x = \lambda(2y)$
 $x = 0$

Like
for param.
e.g. for line.

$$\textcircled{1} \text{ and } \textcircled{2} \Rightarrow \lambda = \underbrace{\frac{y}{18x}}_{\Rightarrow 2y^2 = 18x^2} = \frac{x}{2y} \quad \text{or} \quad (x=0, y=0)$$

$$y^2 = 9x^2$$

Use $\textcircled{4}$

$$\text{If } y^2 = 9x^2,$$

$$\begin{aligned} 9x^2 + y^2 - 4 &= 0 \\ 9x^2 + 9x^2 - 4 &= 0 \\ 18x^2 &= 4 \\ x^2 &= \frac{2}{9} \\ x &= \pm \frac{\sqrt{2}}{3} \end{aligned}$$

$$\text{For both } x = \pm \frac{\sqrt{2}}{3}$$

$$\Rightarrow x^2 = \frac{2}{9}$$

$$\begin{aligned} y^2 &= 9x^2 \\ y^2 &= 9\left(\frac{2}{9}\right) \\ y^2 &= 2 \\ y &= \pm \sqrt{2} \end{aligned}$$

4 candidates: $(\pm \frac{\sqrt{2}}{3}, \pm \sqrt{2})$

\checkmark
All 4 combos
Sign free-form!!

" \pm "s can be ambiguous.

$$\text{If } (x=0, y=0),$$

$$0+0-4=0 \text{ NO!}$$

$(0,0)$ does not
lie on C.
Toss it!

Note: If we had $x=\pm 2, y=3x$

$$x=2 \Rightarrow y=6$$

$$x=-2 \Rightarrow y=-6$$

Only 2 cands.: $(2, 6)$
 $(-2, -6)$

Evaluate f at the candidates.

$$f(x,y) = xy$$

$$f\left(\underbrace{\frac{\sqrt{2}}{3}, \sqrt{2}}_A\right) = \left(\frac{\sqrt{2}}{3}\right)\sqrt{2} = \frac{2}{3} \quad L.A. \text{ Max. value}$$

$$f\left(\underbrace{\frac{\sqrt{2}}{3}, -\sqrt{2}}_B\right) = -\frac{2}{3} \quad L.A. \text{ Min. value}$$

$$f\left(\underbrace{-\frac{\sqrt{2}}{3}, \sqrt{2}}_C\right) = -\frac{2}{3} \quad L.A. \text{ Min. value}$$

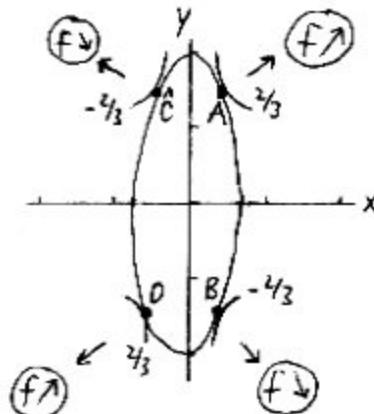
$$f\left(\underbrace{-\frac{\sqrt{2}}{3}, -\sqrt{2}}_D\right) = \frac{2}{3} \quad L.A. \text{ Max value}$$

Why?

Ellipse

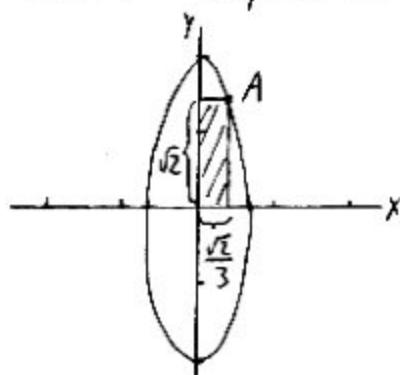
$$\begin{aligned} 9x^2 + y^2 &= 4 \\ \frac{x^2}{\frac{4}{9}} + \frac{y^2}{4} &= 1 \\ \downarrow & \\ a &= 2 \\ b &= \frac{2}{3} \end{aligned}$$

Analyze LCs of f :



Application

Find the dimensions of the rectangle of max. area in QI whose corners are on $(0,0)$, x - and y -axes, and the ellipse $9x^2 + y^2 = 4$.



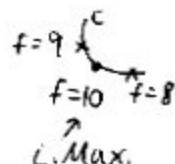
$$\begin{aligned} \frac{\sqrt{2}}{3} \text{ units by } \sqrt{2} \text{ units} \\ \text{Area} = \frac{2}{3} \text{ units}^2 \end{aligned}$$

(E) Strategies for Classifying Candidates

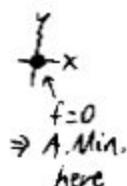
The test is ugly!

- ① Consider the graph (or $C(s)$) of f .
- ② Compare the f values of the candidates.
- ③ If there's only 1 candidate,
try to trace the behavior of f near it
along C , maybe by examining other pts.
on C .

2 pts. just
in case
a SP?

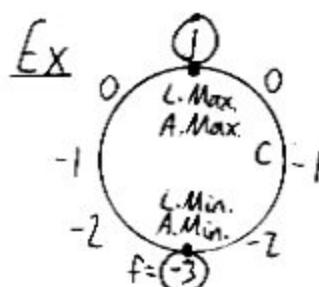


- ④ Examine algebraic properties of f , especially with respect to range. Ex $f(x,y) = x^2 + y^2$



- ⑤ If you're looking for A. Max, Min., examine any endpoints of C .

- ⑥ If f is cont. on a closed curve C (like O^c), and there are only two candidates for local extrema, then one must be a L.Max. (and an A.Max.) and the other must be a L.Min. (and an A.Min.).



Ex

Sign analyses may simplify classification!	By an EVT, there exist. They must occur at L.Max./Min. if C is closed. O^c
-3 vs. +1	

F Strategies for Solving Systems

- ① Eliminate variables one-by-one.
- ② Solve for λ in the $Df = \lambda \vec{V}g$ eqs. and equate.
- ③ Solve for x, y in terms of λ . \Rightarrow Get eq. in λ .
- ④ Multiply both sides of an eq. by something to make elimination easier. (Beware of cases where this something is 0.)

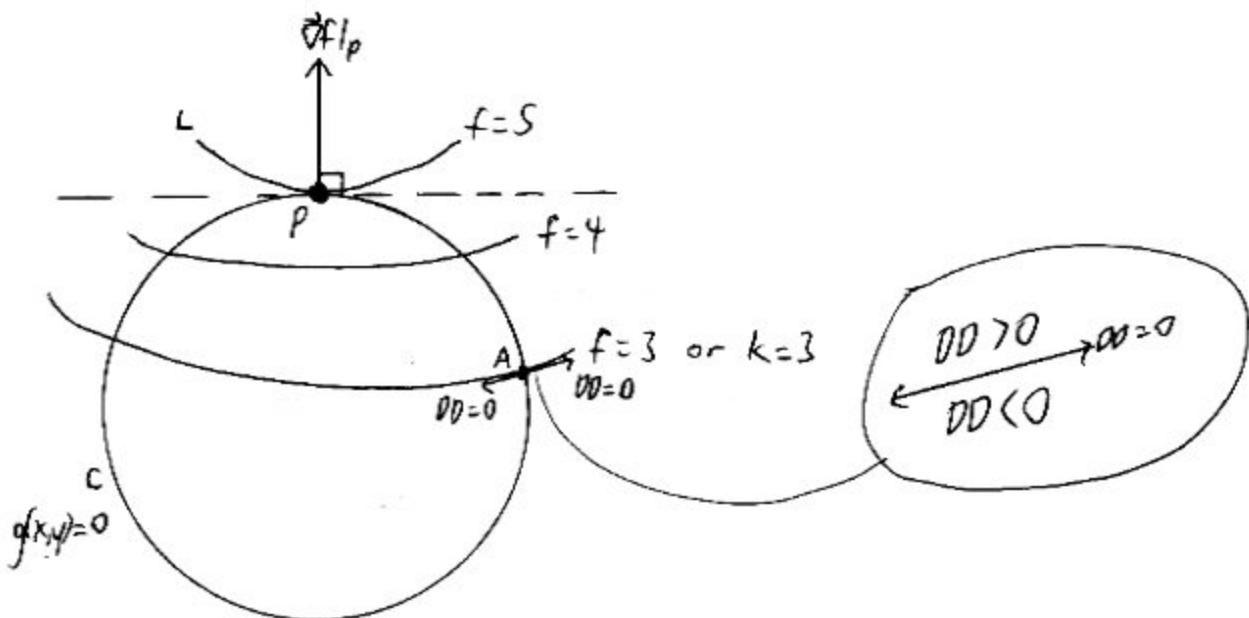
Be mindful of this when canceling!
Factoring is preferable.

⑥ Footnotes

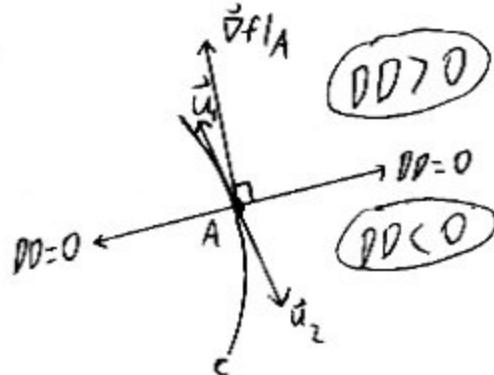
Why focus on where L, C are tangent? (16.9.3)

a level curve of g

a level curve of f



Zoom in:
on A

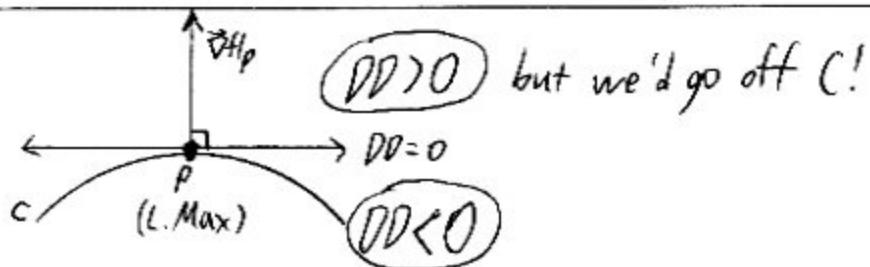


From A, we have
3 choices to try to
max or min f:
 ① Stay at A.
 ② $\uparrow \vec{u}_1$
 ③ $\downarrow \vec{u}_2$

A can't be a L. Max or Min.
To $\uparrow f$, go in the direction of \vec{u}_1 .
 $\downarrow \vec{u}_2$.

If you're at
the North
pole, can you
go further
North?

Zoom in:
on P



DD of f at a point P in the direction of \vec{u} (unit tangent at P to C)

$$= D_{\vec{u}} f(P)$$

$$= (\vec{\nabla} f|_P) \cdot \vec{u}$$

$$= \frac{(\vec{\nabla} f|_P) \cdot \vec{u}}{\|\vec{u}\|} \quad \Leftarrow = 1$$

$$= \text{comp}_{\vec{u}} (\vec{\nabla} f|_P)$$

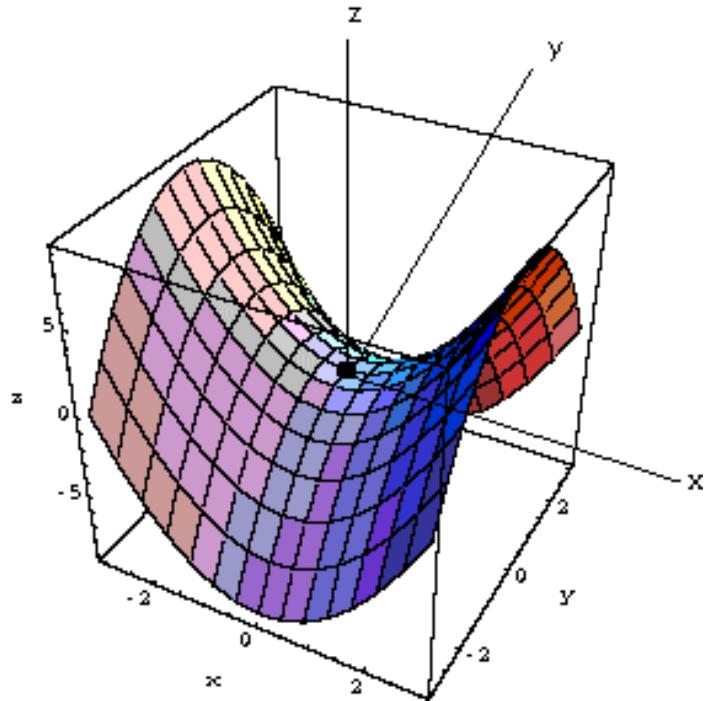
(This = 0) $\Leftrightarrow (\vec{\nabla} f|_P) \perp \vec{u}$

As in Calc I, we care where $DD=0$.

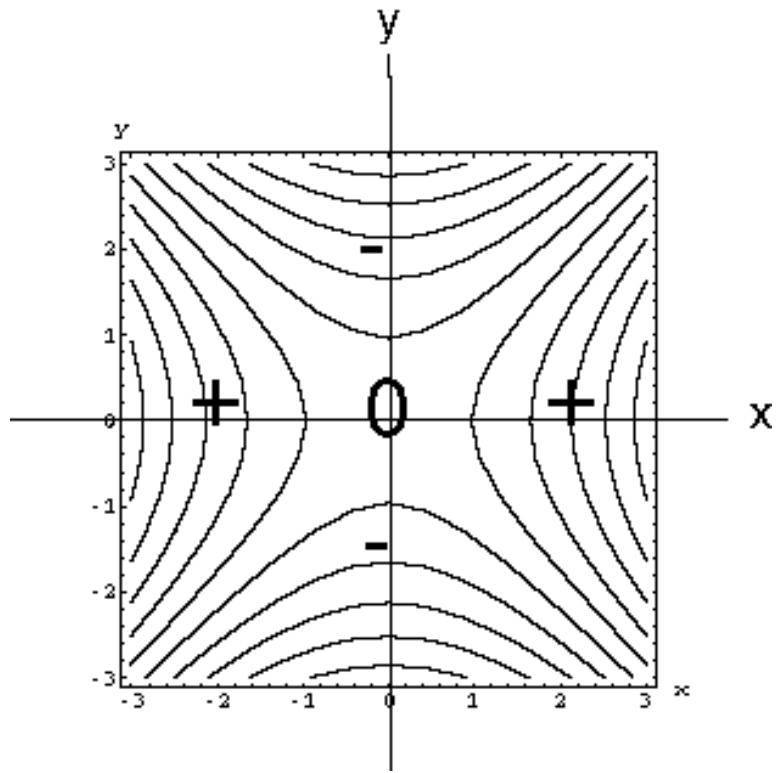
Note: If $\vec{\nabla} f|_P = \vec{0}$, we have $DD=0$ automatically.
In 16.8, we knew that pts. where $\vec{\nabla} f = \vec{0}$ were interesting, anyway.

$$f(x,y) = x^2 - y^2$$

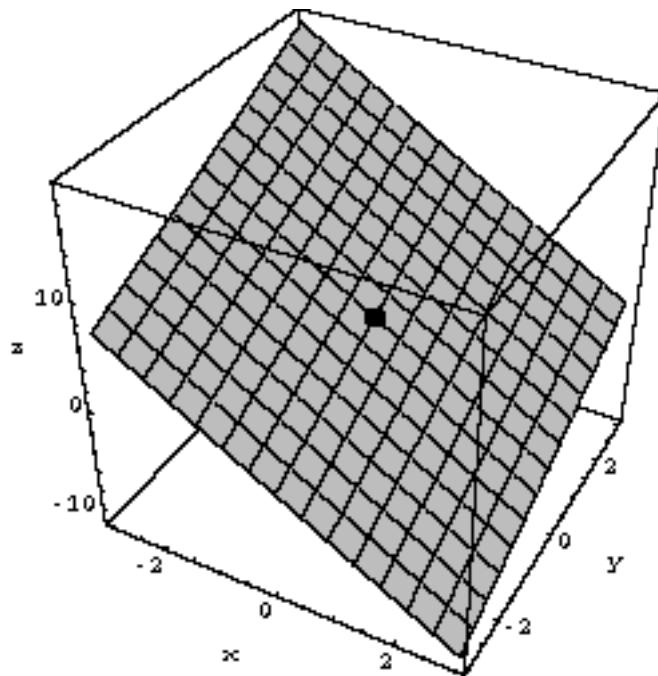
(A Hyperbolic Paraboloid; “Saddle”)



Contour Plot



$$f(x,y) = 4 - 3x + 2y$$



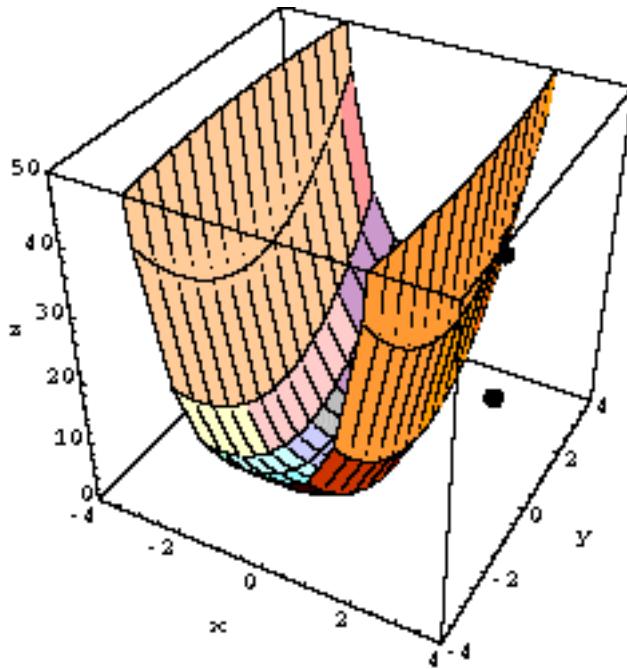
The x -slopes are -3 everywhere (i.e., at all points on the plane); the y -slopes are 2 everywhere.

If we fix any y -value (for example, $y = 0$, which corresponds to the x -axis), we get a cross-sectional line with a slope of -3 in the x -direction.

If we fix any x -value, (for example, $x = 0$, which corresponds to the y -axis) we get a cross-sectional line with a slope of $+2$ in the y -direction.

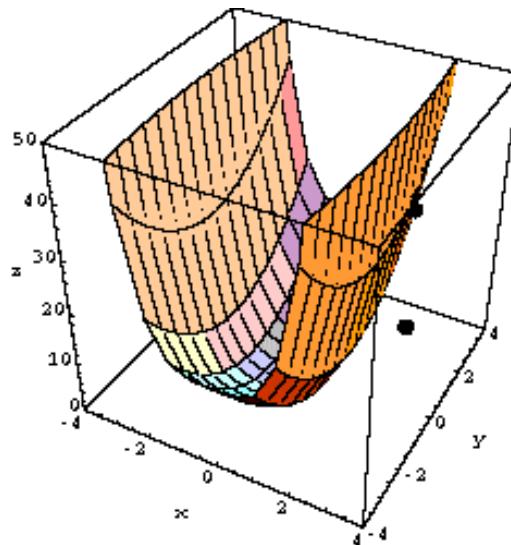
$$f(x,y) \text{ or } z = x^4 + y^2$$

I've plotted the points $(2, 3, 0)$ and $(2, 3, f(2,3) = 25)$, which lies on the surface.

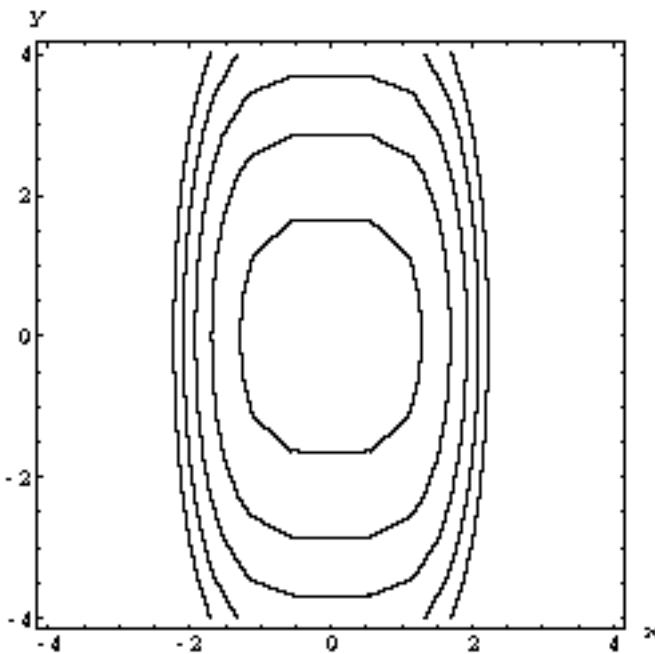


If we fix y to be some value k , we get $f(x,k) = x^4 + k^2 = x^4 + \text{some number}$. Then, the corresponding cross-section is a steep quartic (fourth-degree) curve.

If we fix x to be some value k , we get $f(k,y) = k^4 + y^2 = \text{some number} + y^2$. Then, the corresponding cross-section is a not-as-steep quadratic (second-degree) curve.



Here's the corresponding contour diagram:



It turns out that $f_x(2,3) = 32$ and $f_y(2,3) = 6$; it makes sense that the former is larger, since $f(x,y)$ is a quartic in x but only a quadratic in y . The surface is much steeper in the x -direction than in the y -direction starting from $(x=2, y=3)$. Given that the x - and y -scales are the same in our diagram, it is no wonder that the contours are closer in the x -direction than in the y -direction. In both directions, the contours are getting closer as we move away from $(x=0, y=0)$, indicating that the surface becomes steeper and steeper as we move away from $(x=0, y=0)$.

We have that $\text{grad } f(2,3) = [f_x(2,3)]\mathbf{i} + [f_y(2,3)]\mathbf{j} = 32\mathbf{i} + 6\mathbf{j}$. Note that this gradient pretty much points in the x -direction, with just a little tilt towards the y -direction. This makes sense, since [like a magic compass arrow] the gradient points in the direction where f increases most rapidly from the point you're at.

REVIEW: 16.3-16.916.3: PDS1st-Order Ex

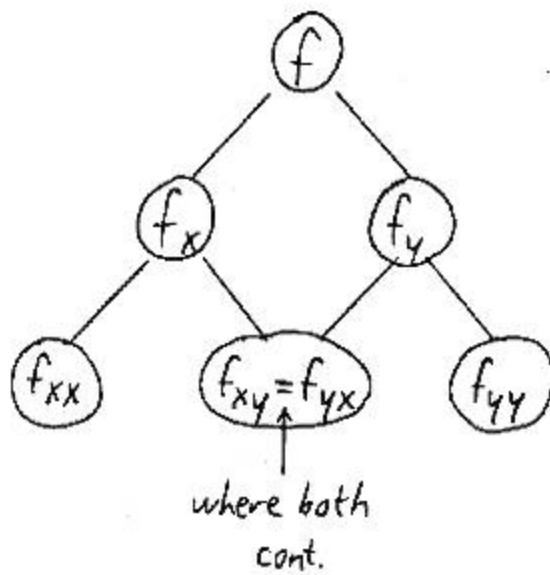
$$f_x(x, y) \text{ or } \frac{\partial f}{\partial x}(x, y)$$

$$= \lim_{h \rightarrow 0} \frac{f(x+h, y) - f(x, y)}{h}$$

To find: Treat y as constant,
Differentiate wrt x .

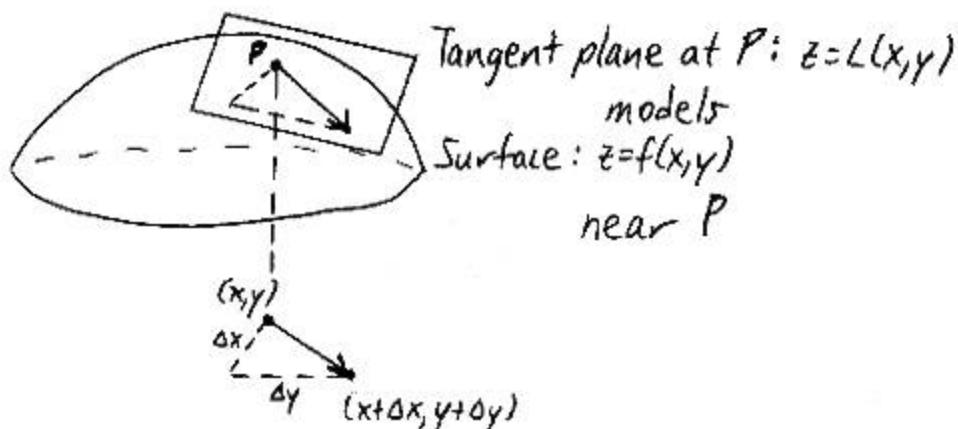
2nd-Order Exs

$$\begin{aligned} f_{xx} &= (f_x)_x = \frac{\partial^2 f}{\partial x^2} \\ f_{xy} &= (f_x)_y = \underline{\frac{\partial^2 f}{\partial y \partial x}} \end{aligned}$$



16.4: INCREMENTS and DIFFERENTIALS

Used to find linear approxs. for f near a seed point $P(x_0, f(x_0))$.



$$\begin{aligned} dx &= \Delta x = \text{new } x - \text{old } x \\ dy &= \Delta y = \text{new } y - \text{old } y \end{aligned}$$

dz = change in z along tangent plane

$$= (x \text{ slope})(x \text{ run}) + (y \text{ slope})(y \text{ run}) \quad \leftarrow \text{Idea: } \text{rise} = (\text{slope})(\text{run})$$

$$= f_x(x,y) \ dx + f_y(x,y) \ dy$$

approximates Δz , the actual change in z along the surface

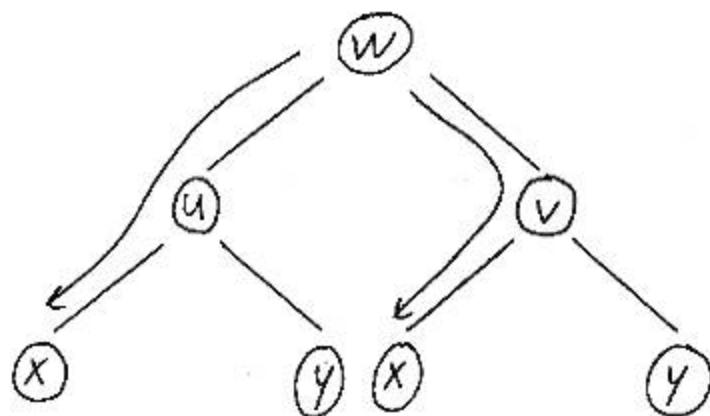
We approx. $f(x + \Delta x, y + \Delta y) = f(x, y) + \Delta z$
 by $L(x + \Delta x, y + \Delta y) = f(x, y) + dz.$

Theory Notes

f_x, f_y cont. on open region $\Rightarrow f(x,y)$ diff'e there
 f diff'e at $(a,b) \Rightarrow f$ cont. there

16.5: CHAIN RULES

Ex

At end, write in terms of x, y

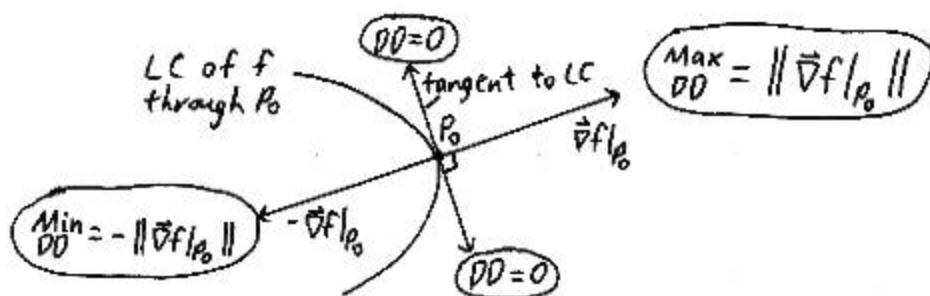
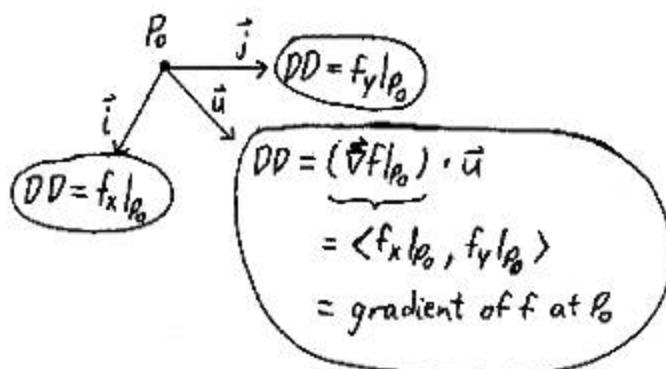
$$\frac{\partial w}{\partial x} = \underbrace{\frac{\partial w}{\partial u} \frac{\partial u}{\partial x}}_{\text{Product along path } w \rightarrow x} + \underbrace{\frac{\partial w}{\partial v} \frac{\partial v}{\partial x}}_{\text{Add these path products.}}$$

If $F(x, y) = 0$ describes a diff'e func. f such that $y = f(x)$,

$$\frac{dy}{dx} = -\frac{F_x}{F_y} \quad (\text{"Negative reciprocal"}) \quad \begin{matrix} F \\ x' \backslash y \end{matrix} \leftarrow \text{treat as indep.}$$

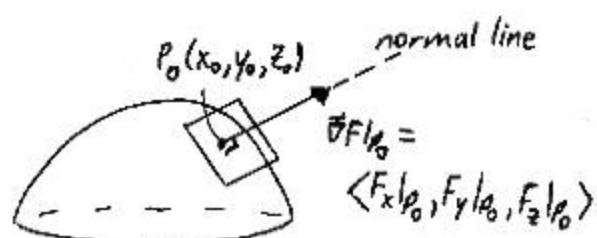
If $F(x, y, z) = 0$ ' $\quad \quad \quad z = f(x, y)$,

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}, \quad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z} \quad \begin{matrix} F \\ x' \backslash y \backslash z \end{matrix} \leftarrow \text{treat as indep.}$$

16.6: DDs // 16.7 $f(x, y)$  $f(x, y, z)$

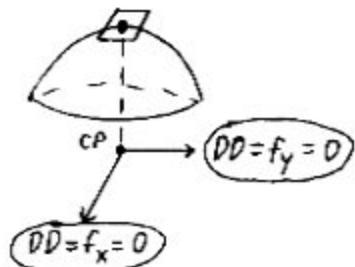
Level curves $\xrightarrow{(LCs)}$ Level surfaces
 Tangent line $\xrightarrow{(TSs)}$ Tangent plane

$$\text{Eq.: } (f_x|_{P_0})(x-x_0) + (f_y|_{P_0})(y-y_0) + (f_z|_{P_0})(z-z_0) = 0$$

Ideas analogous to $f(x, y)$ Ex $\vec{\nabla} f \perp$ (Tangent to LC/LS)

16.8: OPTIMIZATION I

CPs are the only places where L. Max./Min. can occur.



(a, b) is a CP \Leftrightarrow
 ① (a, b) in $\text{Dom}(f)$
 ② $\nabla f(a, b) = \langle f_x(a, b), f_y(a, b) \rangle = \vec{0}$
 or ② DNE

2nd Derivative Test to Classify CPs

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} \quad \left. \begin{array}{l} \text{Assume all} \\ \text{cont.} \end{array} \right\}$$

$$= f_{xx} f_{yy} - (f_{xy})^2$$

At a CP where $\nabla f = \vec{0}$,

① If $D > 0$,

- ①a If f_{xx} (or f_{yy}) < 0 \Rightarrow L. Max.
- ①b $>$ \Rightarrow L. Min.

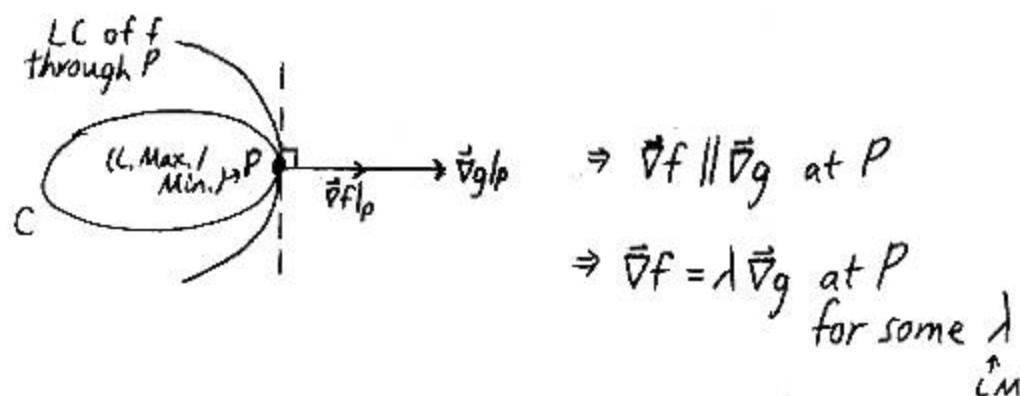
② If $D < 0 \Rightarrow$ Saddle Pt.

③ If $D = 0 \Rightarrow$ No info

16.9: CONSTRAINED OPTIMIZATION - LAGRANGE MULTIPLIERS (LMS)

Ex  ← Find L. or A. Max./Min. [Pts.] of f along [the image of] C .

$$C: g(x, y) = 0$$



Solve $\begin{cases} \vec{\nabla}f(x, y) = \lambda \vec{\nabla}g(x, y) \\ g(x, y) = 0 \end{cases}$ for (x, y, λ)

↑
 can differ among
 (x, y) candidates
 don't have to find

Evaluate f at the candidates, and compare.

↑
 See Strategies for Classifying, Solving Systems.