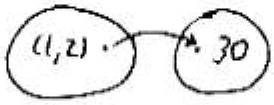
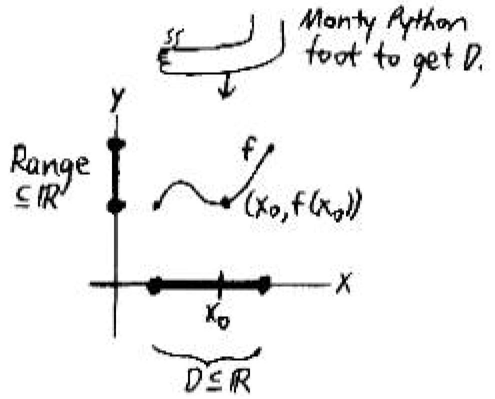
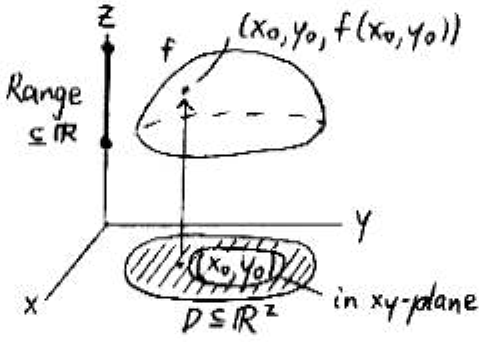
CH. 16

16.1: FUNCTIONS OF SEVERAL VARIABLES

(A) Intro

How to Ace

Ch. 15	vs. Now
Domain $D \subseteq \mathbb{R}$ (is a subset of)	$D \subseteq \mathbb{R}^n$
VVF $\vec{r} : D \longrightarrow V_n$ (or $\mathbb{R}^n$)	$f : D \to \mathbb{R}$
	
$\vec{r}(1) = \langle 10, 20 \rangle$	$f(1,2) = 30$

Calc I	vs. Now
$y = f(x)$	$z = f(x,y)$; z: dependent variable; x, y: independent variables
$x \Rightarrow \boxed{f} \Rightarrow y$	$x, y \Rightarrow \boxed{f} \Rightarrow z$
	

It is usually assumed that
$D = \{(x,y) \mid f(x,y) \text{ is defined (as a real \#)}\}$

How would you represent $w = f(x,y,z)$?

Ex Let $f(x,y,z) = \dfrac{\sqrt{4-z}}{x+3} + \ln(x+y) - \tan z$.

ⓐ Find D.

$4 - z \geq 0$

$z \leq 4$,

$x + 3 \neq 0$

$x \neq -3$

$x + y > 0$

tan as slope-
where undefined?

$z \neq \frac{\pi}{2} + \pi n$ (n integer)

$$\boxed{D = \{(x,y,z) \mid x \neq -3,\ x+y > 0,\ z \leq 4, \text{ and } z \neq \tfrac{\pi}{2} + \pi n \text{ (n integer)}\}}$$

ⓑ Find $f(1,2,3)$.

$$= \frac{\sqrt{4-3}}{1+3} + \ln(1+2) - \tan 3$$

$$= \boxed{\tfrac{1}{4} + \ln 3 - \tan 3}$$

Ⓑ Level Curves (LCs)

Idea

Graph $z = f(x,y)$ in $\mathbb{R}^3$.

Take traces in planes $z=k$.
(The k-values are preferably evenly spread out.)
(Think: Different colors on different transparencies.)

Andromeda strain

Project them onto the xy-plane, and indicate their k-values.
(Think: Overlaying the transparencies. foot)

In Practice

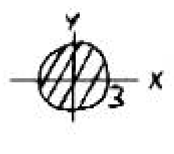
The LC of f for $z=k$ is the graph of
$k = f(x,y)$ in $\mathbb{R}^2$.
(Replace z with k.)

Ex $f(x,y) = \sqrt{9-x^2-y^2}$

Find D

$$9 - x^2 - y^2 \geq 0$$
$$9 \geq x^2 + y^2$$
$$x^2 + y^2 \leq 9$$

$$\boxed{D = \{(x,y) \mid x^2+y^2 \leq 9\}}$$



Idea of LCs

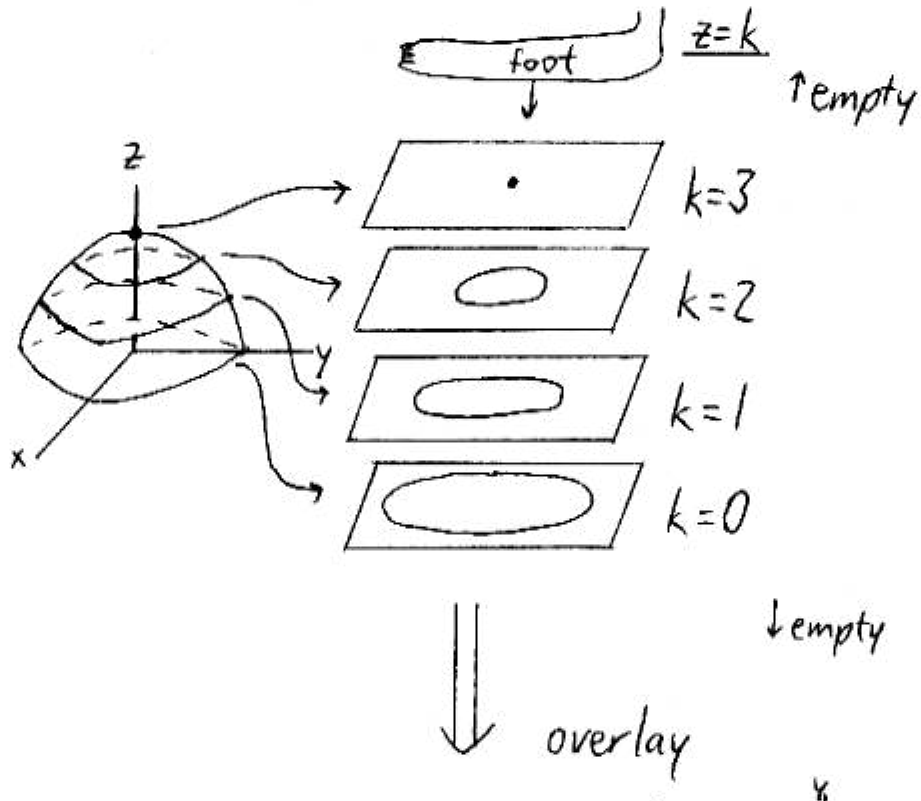
Graph $z=\sqrt{9-x^2-y^2}$

$$z^2 = 9-x^2-y^2 \;,\quad z\ge 0$$

$$x^2+y^2+z^2=9 \;,\quad z\ge 0$$

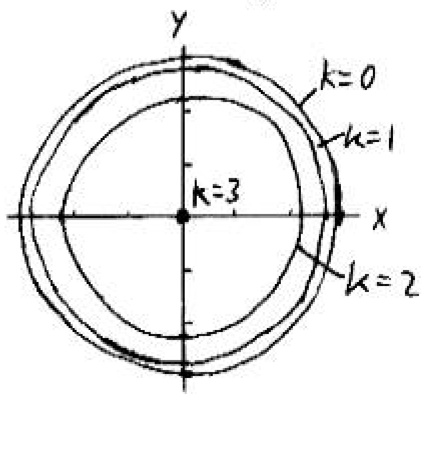
Upper half of the sphere of radius 3 centered at $(0,0,0)$.

Sideshow Bob?



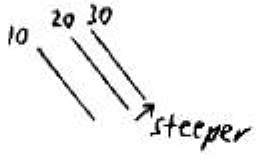
x — y ⟹ y — x in $\mathbb{R}^2$

Radii:
3
$\sqrt{8}\approx 2.8$
$\sqrt{5}\approx 2.2$
0



You can try to reconstruct the surface from its LCs.
Lift/drop a LC by $|k|$ units.

If the k-values are evenly spaced out, usually,
closer LCs $\Longrightarrow$ faster $\nearrow$ in f-value.
"steeper climb"



In Practice

LC for $z=k$:

$$k = \sqrt{9-x^2-y^2}$$
$$k^2 = 9-x^2-y^2 \ , \ k \geq 0$$
$$x^2+y^2 = 9-k^2 \ , \ k \geq 0$$

If $k < 0 \Longrightarrow$ empty

If $0 \leq k < 3 \longrightarrow$ circle of radius $\sqrt{9-k^2}$ centered at $(0,0)$

If $k=3 \underset{x^2+y^2=0}{\Longrightarrow} (0,0)$ only

If $k > 3 \underset{9-k^2<0}{\Longrightarrow}$ empty

Ⓒ Applications

Used in topographic/contour maps: Mountain climbing (bird's-eye view), Lake depths
Weather map - LCs are isotherms.
same temperature

(D) Level Surfaces (LSs)

Let $w = f(x,y,z)$ be a temperature function, say.

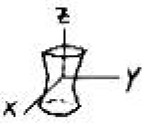
The LS for $w=k$ is often a 3D surface in $\mathbb{R}^3$ consisting of points with the same temperature.

Ex 6, p.799
axis: y-axis

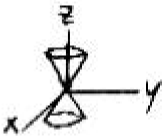
Ex (#40) $\underbrace{f(x,y,z)}_{=w} = x^2+y^2-z^2$

LS for $w=k$

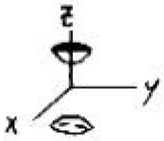
$k=1$: $1 = x^2+y^2-z^2$ Hyp. of 1 sheet



$k=0$: $0 = x^2+y^2-z^2$
$z^2 = x^2+y^2$ Cone



$k=-1$: $-1 = x^2+y^2-z^2$
$z^2 = x^2+y^2+1$ Hyp. of 2 sheets



↑
All with axes
along the z-axis.
No shared points!
LCs never intersect, and
LSs 〃 〃.

Think: Morphing as k changes.