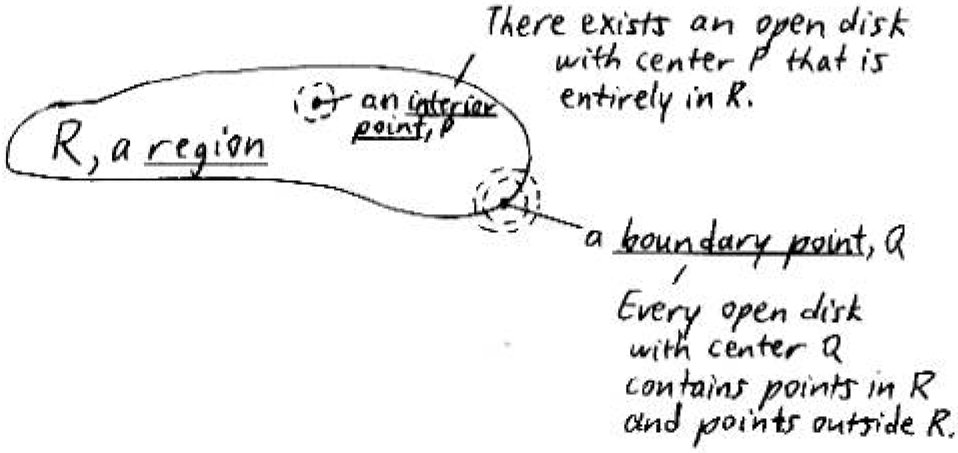
16.2: LIMITS and CONTINUITY

(A) Point Set Topology

closed disk   open disk



The Village

For limit and continuity analyses, always stay in R!

R is <u>open</u> $\Longleftrightarrow$ Every point in R is an interior point.

R is <u>closed</u> $\Longleftrightarrow$ R includes all its boundary points.

(B) Notation

Relate to (A)
Erdős ϵ = kid

$\forall$ - for every ϵ - epsilon

$\exists$ - there exists δ - delta (lowercase)

(C) Defining Limits - Calc I (See Section 2.2 Notes.)

"$\lim_{x \to a} f(x) = L$"

(or "$f(x) \to L$ as $x \to a$")

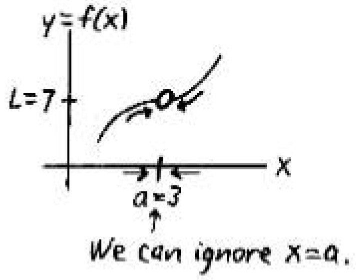
means that

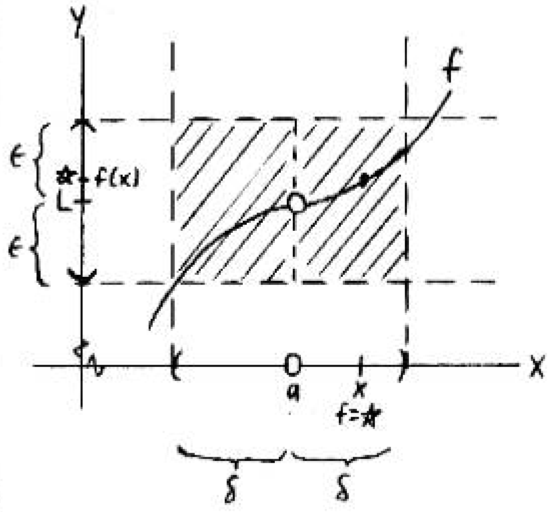
$\forall \epsilon > 0, \exists \delta > 0$ such that

<u>if</u> $\underbrace{0 <}_{x \neq a} \underbrace{|x-a|}_{\text{distance between } x,\, a} < \delta$

<u>then</u> $\underbrace{|f(x) - L|}_{\text{distance between } f(x),\, L} < \epsilon$

<u>Ex</u> $\lim_{x \to 3} f(x) = 7$

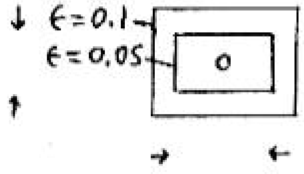




Given ϵ, find a δ so that the box traps the graph $(x \neq a)$.

As ϵ shrinks, δ tends to shrink.

"Walls closing in" / "Zooming in"



ⓓ Defining Limits – Calc III

"$\lim_{(x,y)\to(a,b)} f(x,y) = L$"

(or "$f(x,y) \to L$ as $(x,y) \to (a,b)$")

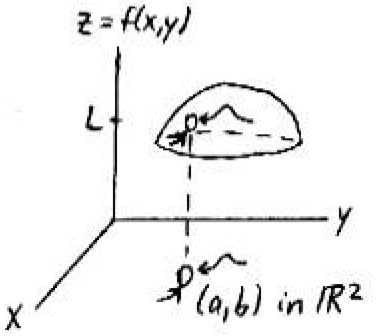
means that

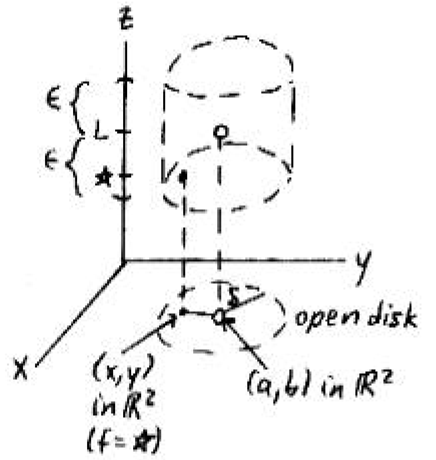
$\forall \epsilon > 0, \exists \delta > 0$, such that

<u>if</u> $\underbrace{0 <}_{(x,y)\neq(a,b)} \underbrace{\sqrt{(x-a)^2 + (y-b)^2}}_{\text{distance between }(x,y)\text{ and }(a,b)} < \delta$

<u>then</u> $\underbrace{|f(x,y) - L|}_{\text{distance between } f(x,y) \text{ and } L} < \epsilon$

<u>Ex</u>



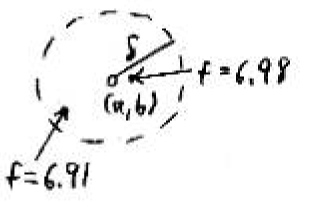


Given ϵ, find a δ so that the cylinder traps the surface $((x,y) \neq (a,b))$.

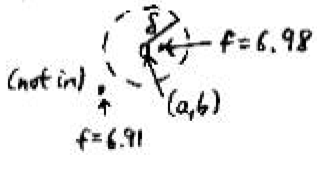
As ϵ shrinks, δ tends to shrink.

<u>Ex</u> $L = 7$

$\epsilon = 0.1$



$\epsilon = 0.05$



Ⓔ Computing Limits

Basic Limit Laws from pp. 60-64 (2.3) extend naturally.

unwritten in class

The limit of a(n) sum	= The sum of the limits (if they exist)
difference	difference
product	product
quotient	quotient (if denom. $\neq 0$)
[nice] power	power
odd root	odd root
even root	even root (if radicand > 0)

Ex $\displaystyle\lim_{(x,y)\to(1,-2)} \underbrace{\frac{4y+\sqrt{x^2+y^2}}{x^3y^2-1}}$ ← polynomial in x and y

$f(x,y)$ is algebraic (rational but $\sqrt{\ }$ OK)
If there's no problem with $\frac{\ }{0}$ or $\sqrt[\text{even}]{\le 0}$,
use direct sub to find the limit.

$$= \frac{4(-2)+\sqrt{(1)^2+(-2)^2}}{(1)^3(-2)^2-1}$$

$$= \boxed{\frac{\sqrt{5}-8}{3}}$$

If $\frac{0}{0}$ limit form $\Longrightarrow$ Factor and Cancel?
Rationalize the denom. or numer.?
L'Hôpital's Rule? (If one variable...)

When Animals Attack

(F) When Limits Do Not Exist (DNE)

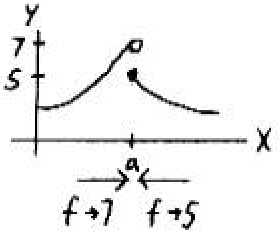
includes $\infty, -\infty$

Calc I

If $\lim\limits_{x \to a^-} f(x) \neq \lim\limits_{x \to a^+} f(x)$, then $\lim\limits_{x \to a} f(x)$ DNE.

or if either DNE ⇗

Ex



Calc III

Two-Path Rule

If two different paths to (a,b) yield different limit values for f, then $\lim\limits_{(x,y)\to(a,b)} f(x,y)$ DNE.

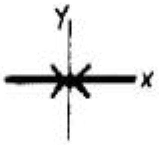
(Also, if any path yields an undefined limit ⇈)

Ex

$\Rightarrow \lim\limits_{(x,y)\to(a,b)} f(x,y)$ DNE

<u>Ex</u> Show $\overbrace{\lim\limits_{(x,y)\to(0,0)} \dfrac{2x^2+3y^2}{4x^2-5xy+6y^2}}^{\circledast}$ DNE.

<u>(i) $(x,y)\to(0,0)$ along the x-axis $(y=0)$</u>

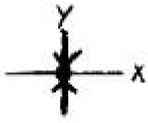


$$\lim_{(x,y)\to(0,0)} \frac{2x^2+3(0)^2}{4x^2-5x(0)+6(0)^2}$$

$$= \lim_{(x,y)\to(0,0)} \frac{2x^2}{4x^2}$$

$$= \frac{1}{2}$$

<u>(ii) $(x,y)\to(0,0)$ along the y-axis $(x=0)$</u>



$$\lim_{(x,y)\to(0,0)} \frac{2(0)^2+3y^2}{4(0)^2-5(0)y+6y^2}$$

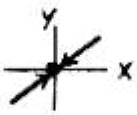
$$= \lim_{(x,y)\to(0,0)} \frac{3y^2}{6y^2}$$

$$= \frac{1}{2}$$

<u>WARNING</u>: This does <u>not</u> prove that $\circledast = \frac{1}{2}$.

Waste of time, ☹ as it turns out

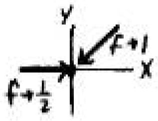
<u>(iii) $(x,y)\to(0,0)$ along the line $y=x$</u> (say)



$$\lim_{(x,y)\to(0,0)} \frac{2x^2+3(x)^2}{4x^2-5x(x)+6(x)^2}$$

$$= \lim_{(x,y)\to(0,0)} \frac{5x^2}{5x^2}$$

$$= 1$$



By the Two-Path Rule, Ⓐ DNE.

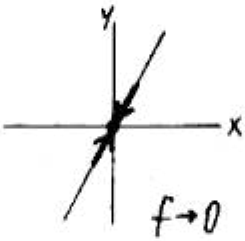
Ex 4 (p. 809)

$$\lim_{(x,y)\to(0,0)} \overbrace{\frac{x^2 y}{x^4+y^2}}^{f(x,y)} \text{ DNE}$$

Book literally throws you a curve

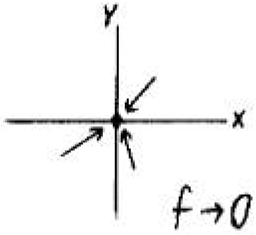
Idea: As $m \to 0$ ($\theta \to 0$ or π), f changes more rapidly ($\lim \frac{mx}{x^2+m^2}$) ↓ along the line

Along $y=mx$ ($m \neq 0$)



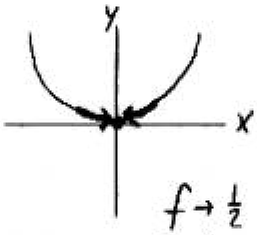
$f \to 0$ both ways

Let m vary:



$f \to 0$

Along $y=x^2$

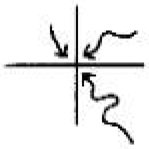


$f \to \frac{1}{2}$ both ways

Up to 11

Ⓖ Using Polar Coordinates to Find Limits

$$\lim_{(x,y)\to(0,0)} \frac{x^4}{x^2+y^2} \quad \left(\frac{0}{0}\right) = \lim_{r\to 0} \frac{(r\cos\theta)^4}{r^2}$$



Any way you approach $(0,0)$
$\Rightarrow r \to 0$

$$= \lim_{r\to 0} \frac{r^4\cos^4\theta}{r^2}$$

$$= \lim_{r\to 0} r^2 \underbrace{\cos^4\theta}_{\text{in } [-1,1] \text{ always}}$$

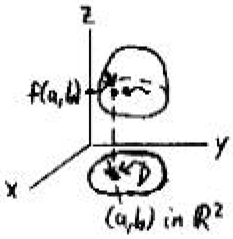
Up to 23

$$= \boxed{0} \qquad (\theta \text{ is irrelevant!})$$

Ⓗ Continuity

f is <u>continuous</u> at an interior point (a,b) of its domain, D, if
$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$ (and these exist).

How to Ace: Mountain hazards at disconts.



(For a boundary point, make sure paths stay in D.)

f is cont. on $D \leftrightarrow f$ is cont. at every (a,b) in D.

Rules from pp. 80–83 (2.5) extend naturally.
If f, g cont. at $(a,b) \Rightarrow$ so are $f \pm g$ and $\frac{f}{g} \Leftarrow$ if $g(a,b) \neq 0$.
If g cont. at (a,b), f cont. at $g(a,b) \Rightarrow f \circ g$ cont. at (a,b).

Domain of $f \subseteq \mathbb{R}$