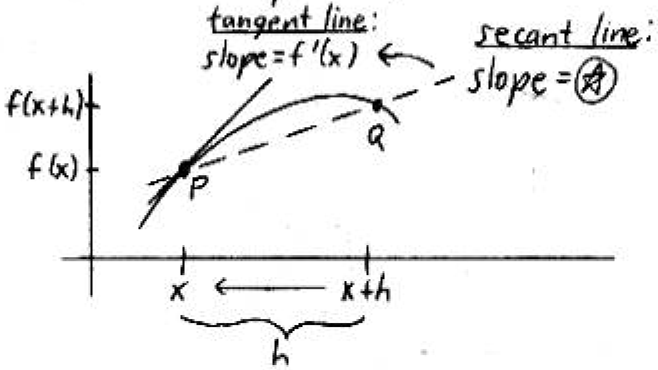
Larson: 1730-60
Euler, Jean D'Alembert

16.3: PARTIAL DERIVATIVES (PDs)

A) Review Calc I: $y=f(x)$



$$\boxed{f'(x) = \lim_{h\to 0} \underbrace{\frac{f(x+h)-f(x)}{h}}_{☆}}$$

or fix $x=x_0$

B) Now, Calc III: $z=f(x,y)$

Δ, δ

"del"

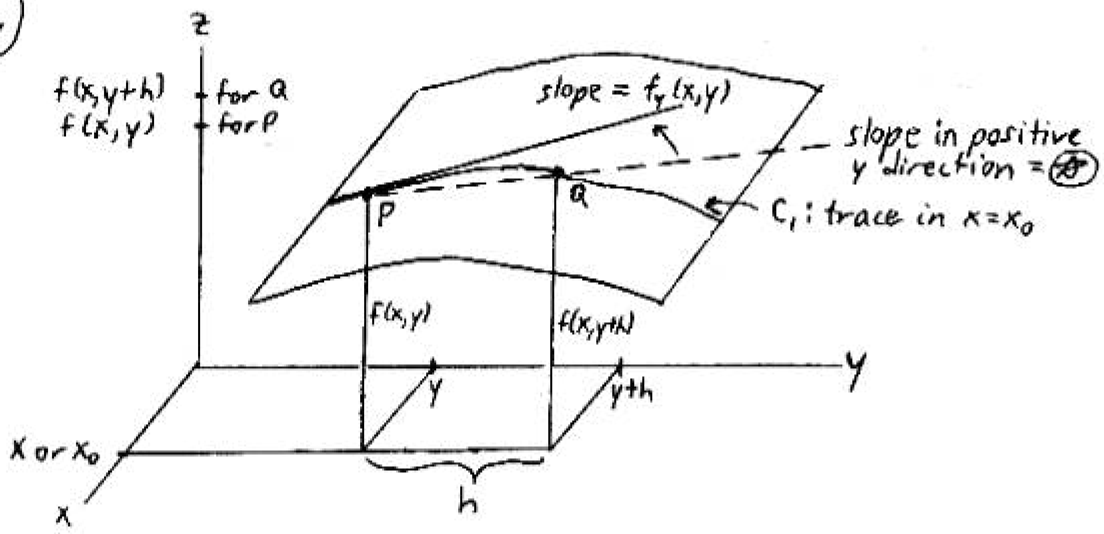
$f_x = \frac{\partial f}{\partial x}$ = the partial derivative of f with respect to x (wrt)

$f_y = \frac{\partial f}{\partial y}$ = " " y

Leibniz notation

He had d, not ∂.

f_y



$$\boxed{f_y(x,y) = \lim_{h\to 0} \underbrace{\frac{f(x,y+h)-f(x,y)}{h}}_{★}}$$

= slope of tangent line to C_1 at P

= instantaneous rate of change of f wrt y at P

We treat x as constant, and we differentiate $f(x,y)$ wrt y.

f_x

$$\boxed{f_x(x,y) = \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h}}$$

We treat y as constant, and we differentiate $f(x,y)$ wrt x.

© Exs

D_x rules from Calc I extend naturally.

Chain Rule Ex

Books omit ( )
How to Ace →

$$\underbrace{\frac{\partial}{\partial x}}_{\substack{\partial_x \\ D_x}}(\sin u) = (\cos u)\left(\frac{\partial u}{\partial x}\right)$$

Ex $f(x,y) = xy^3 + \ln(2x - 3y^2)$

ⓐ Find $f_x(x,y)$

$$f(x,y) = x\underbrace{y^3}_{\substack{\text{"\#"} \\ \text{treat as} \\ \text{constant}}} + \ln(2x - 3\underbrace{y^2}_{\#})$$

← Would ln properties help? Not here...

Calc I: $D(\ln \star) = \frac{1}{\star} \cdot D(\star)$

$$f_x(x,y) = \underbrace{y^3}_{\text{Think: } D_x(x \cdot 7) = 7} + \frac{1}{2x-3y^2} \cdot \underbrace{D_x(2x - 3\underbrace{y^2}_{\#})}_{=2}$$

$$= \boxed{y^3 + \frac{2}{2x-3y^2}}$$

(b) Find $f_y(x,y)$

$$f(x,y) = \underbrace{x}_{\#} y^3 + \ln(\underbrace{2x}_{\#} - 3y^2)$$

$$f_y(x,y) = \underbrace{x(3y^2)} + \frac{1}{2x-3y^2} \cdot D_y(\underbrace{2x}_{\#} - 3y^2)$$

7-factor doesn't disappear

Think: $D_y(7y^3) = 7(3y^2)$

$$= 3xy^2 + \frac{1}{2x-3y^2} \cdot (-6y)$$

$$= \boxed{3xy^2 - \frac{6y}{2x-3y^2}}$$

Deal with later: $D(e^{*}) = e^{*} D(*)$

Ex $f(x,y,z) = \underbrace{y}_{\#} e^{\overbrace{xy+yz}}$. Find f_x and $f_x(0,3,4)$.

$$f_x(x,y,z) = ye^{xy+yz} \cdot \underbrace{D_x(x\underbrace{y}_{\#} + \underbrace{yz}_{\#})}_{\begin{subarray}{l} = y + 0 \\ = y \end{subarray}}$$

$$= \boxed{y^2 e^{xy+yz}}$$

$$f_x(0,3,4) = (3)^2 e^{(0)(3)+(3)(4)}$$

$$= \boxed{9e^{12}}$$

① 2^{nd}-Order PDs

$$f_{xx} = (f_x)_x = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} \text{ or } \underbrace{\frac{\partial^2}{\partial x^2}}_{\text{operator}} f$$

$$f_{yy} = (f_y)_y = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} \text{ or } \frac{\partial^2}{\partial y^2} f$$

$\Rightarrow$ Concavity in x,y directions

Mixed Partials

Larson: Start w/variable nearest f

$$f_{\overrightarrow{xy}} = (f_x)_y = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}$$

$$f_{\overrightarrow{yx}} = (f_y)_x = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y}$$

These are equal where both are continuous.

These extend naturally to higher-order PDs.

Ex

$f(x,y) = \boxed{(3x+y^2)^5}$

D_x: f_x: $5(3x+y^2)^4(3) = \boxed{15(3x+y^2)^4}$

D_y: f_y: $5(3x+y^2)^4(2y) = \boxed{10y(3x+y^2)^4}$

D_x: f_{xx}: $60(3x+y^2)^3(3) = \boxed{180(3x+y^2)^3}$

D_y: f_{xy}: $60(3x+y^2)^3(2y) = \boxed{120y(3x+y^2)^3}$

D_x: f_{yx}: $10y \cdot 4(3x+y^2)^3(3) = \boxed{120y(3x+y^2)^3}$

D_y: f_{yy} Product Rule

f_{yy}: $[D_y(10y)]\cdot(3x+y^2)^4 + (10y)\cdot D_y[(3x+y^2)^4]$

$= 10(3x+y^2)^4 + (10y)\cdot 4(3x+y^2)^3(2y)$

$= \boxed{10(3x+y^2)^4 + 80y^2(3x+y^2)^3}$