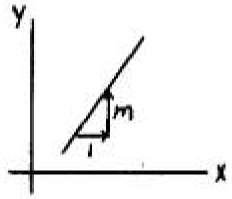
16.4: INCREMENTS and DIFFERENTIALS

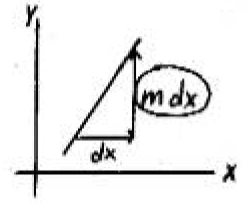
Not in How to Ace

$\Delta x, \Delta y, \Delta z, \Delta w$ $\qquad$ dx, dy, dz, dw

Ⓐ Interpreting Slope

$f(x) = mx + b$



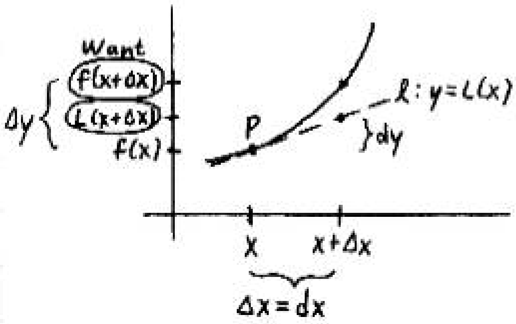


$dx < 0$ ←

run 1 $\Rightarrow$ rise m $\qquad$ run dx $\Rightarrow$ rise $m\,dx$

$$\boxed{\text{slope} = \frac{\text{rise}}{\text{run}} \Rightarrow \text{rise} = (\text{slope})(\text{run}) = f'(x)\,dx}$$

Ⓑ Review Calc I: $y = f(x)$



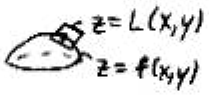
Find $L(x+\Delta x)$, a linear approximation for $f(x+\Delta x)$ based on a "seed" point $P(x, f(x))$ and its tangent line, ℓ.

Approx. $f(x+\Delta x) = f(x) + \Delta y$
by $\quad L(x+\Delta x) = f(x) + dy$

where dy = rise along ℓ as $x \to x + \Delta x$
$= f'(x)\,dx$

(C) Now, Calc III: $z = f(x,y)$

Based on a "seed" point $P(x, y, f(x,y))$ and its $\underbrace{\text{tangent plane}}_{\text{best linear model around } P}$ to the surface (graph of f),



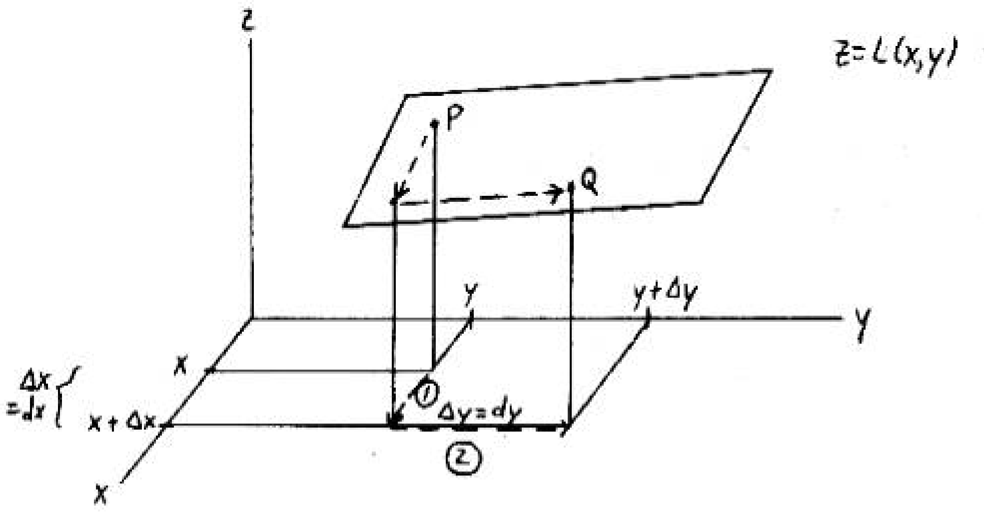
$$\text{approx. } f(x+\Delta x, y+\Delta y) = f(x,y) + \underbrace{\Delta z}$$

actual change in z along surface as "shadow" $(x,y) \to (x+\Delta x, y+\Delta y)$

$$\text{by } \quad L(x+\Delta x, y+\Delta y) = f(x,y) + \underbrace{dz}$$

change in z along tangent plane

D What is dz?



dz = total differential of z

= change in z from P to Q

= (change in z from Stage ①) +
(" " ②)

= (slope of tangent plane in x-direction)(run in x) +
(" y ")(" y)

= $f_x(x,y)\,dx$ +
$f_y(x,y)\,dy$

If $w = f(x,y,z)$,

$$dw = f_x\,dx + f_y\,dy + f_z\,dz$$

Ⓔ Ex

$f(x,y) = x^2 + 3xy^2$

Use the fact that $f(1,2) = 13$ to find a linear approx. for $f(0.98, 2.03)$.

Sol'n

$dx = \text{new } x - \text{old } x$
$= -0.02$

$$(1,2) \xrightarrow{dx = -0.02} (0.98, 2.03)$$

$$(1,2) \xrightarrow{dy = 0.03} (0.98, 2.03)$$

$$dz = [f_x(1,2)]\,dx + [f_y(1,2)]\,dy$$

$$f_x(x,y) = 2x + 3y^2$$
$$f_x(1,2) = 2(1) + 3(2)^2$$
$$= 14$$

$$f_y(x,y) = 3x(2y)$$
$$= 6xy$$
$$f_y(1,2) = 6(1)(2)$$
$$= 12$$

Once you do this work, you can quickly approx. $f(x,y)$ where $x \approx 1$ and $y \approx 2$. Multiple f evaluations?

$$= (14)(-0.02) + (12)(0.03)$$
$$= 0.08$$

$$L(0.98, 2.03) = f(1,2) + dz$$
$$= 13 + 0.08$$
$$= \boxed{13.08}$$

Exact: 13.075846

$\Delta z = 0.075846$

Ⓕ Applications

Given: Limited f info in a table

$\Rightarrow$ Estimate f_x, f_y at a "seed" in the table

$\Rightarrow$ Perform linear interpolations for f using differentials (near the seed)

Table:

$x \backslash y$	0	100	200
0	36	38	42
10	40	43	47
20	45	48	51

Ⓖ Theory

Advanced Note

f is <u>differentiable</u> at (a,b) if we can write

$$\Delta z = f_x(a,b)\,\Delta x + f_y(a,b)\,\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

($\varepsilon_1, \varepsilon_2$:) functions of $\Delta x, \Delta y$ $\to 0$ as $\Delta x \to 0$, $\Delta y \to 0$

Stewart: f_x, f_y $\exists$ but f, f_x, f_y not cont. at $(0,0)$

$\downarrow$

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

If f_x, f_y cont. on an open region R $\Rightarrow$ f is differentiable (diff'e) on R.

$\Rightarrow$ Linear approxs. tend to get better as $\Delta x \to 0$ and $\Delta y \to 0$.

If f is diff'e at (a,b) $\Rightarrow$ f is cont. there.