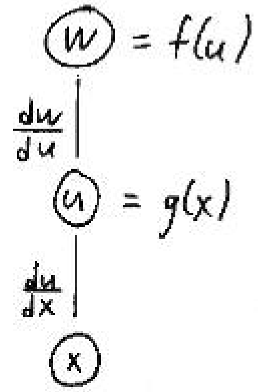
16.5: CHAIN RULES

(A) Intro

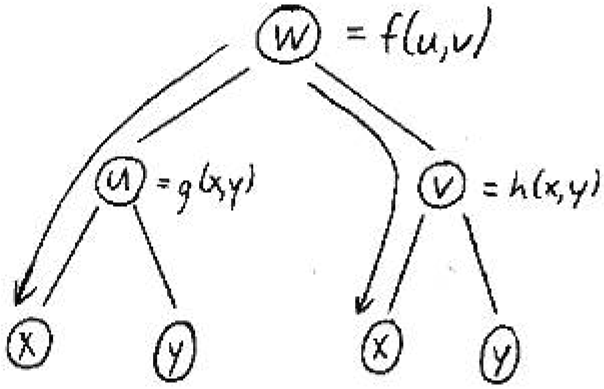
Assume funcs. are diff'e where we care.

Calc I

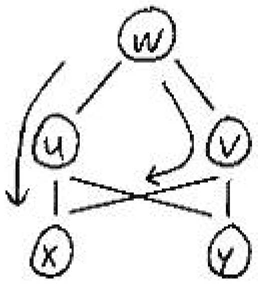


$$\frac{dw}{dx} = \frac{dw}{du}\,\frac{du}{dx}$$

Calc III



or

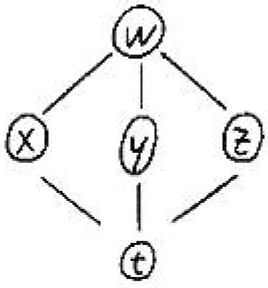


Plinko model:

For $\frac{\partial w}{\partial x}$,
take products along paths from w to x, and add them.
(due to 16.4 ideas)

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial x}$$

Y Y Y

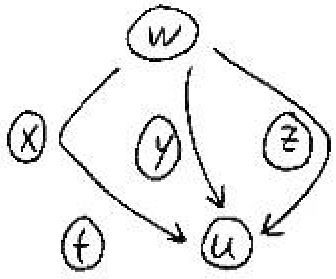


$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

Use "d" if there's only 1 variable at this level

Ex Find $\frac{\partial w}{\partial u}$ if $w = 3x^2 + e^{4y} - x \ln z$,
where $x = \sin(tu)$, $y = t^3 + u$, and $z = u^4$.

Sol'n



$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u}$$

$$\begin{aligned} &= D_x\,(3x^2 + e^{4y} - x \ln z)\cdot D_u\,[\sin(tu)] \\ &\quad + D_y\,(\quad \text{''} \quad)\cdot D_u\,[t^3+u] \\ &\quad + D_z\,(\quad \text{''} \quad)\cdot D_u\,[u^4] \end{aligned}$$

$$\begin{aligned} &= (6x - \ln z)\cdot[t\cos(tu)] \\ &\quad + (4e^{4y})\cdot(1) \\ &\quad + (-x\cdot \tfrac{1}{z})\cdot(4u^3) \end{aligned}$$

$$= (6x - \ln z)\, t\cos(tu) + 4e^{4y} - \frac{4xu^3}{z}$$

Sub into x, y, z.

$$= [6\sin(tu) - \ln(u^4)]\, t\cos(tu)$$
$$+ 4e^{4(t^3+u)}$$
$$- \frac{[4\sin(tu)]\cancel{u^3}}{\cancel{u^4}\, u}$$

$$= \boxed{[6\sin(tu) - 4\ln|u|]\, t\cos(tu) + 4e^{4(t^3+u)} - \frac{4\sin(tu)}{u}}$$

Up to 13

In many problems,
easier than if we had subbed into x, y, z immediately!

(B) Getting Derivatives from Implicit Functions

(Shortcuts to 3.7 Method)

If $F(x,y) = 0$ determines a diff'e func. f
such that $y = f(x)$, then

$$\boxed{\frac{dy}{dx} = -\frac{F_x}{F_y}}$$

Think "negative reciprocal."
Proof based on Chain Rule.

If $F(x,y,z) = 0$ " " such that $z = f(x,y)$, then

$$\boxed{\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}, \quad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}}$$

$\underline{Ex}$ Find $\frac{\partial z}{\partial x}$ if $\underbrace{x^2 z + \tan(yz)}_{= F(x,y,z)} = 0$ (←Can't solve for z.)

$\underline{\text{Sol'n}}$

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}$$

$$= \boxed{-\frac{2xz}{x^2 + y\sec^2(yz)}}$$

$\underline{\text{Section 3.7 Method}}$ (for comparison :))

$$z = f(x,y)$$

$$\underbrace{D_x(x^2 z)}_{\text{Use Product Rule!}} + D_x[\tan(yz)] = 0$$

$$2xz + x^2 \frac{\partial z}{\partial x} + [\sec^2(yz)][y \frac{\partial z}{\partial x}] = 0$$

$$= -2xz$$

$$\frac{\partial z}{\partial x}[x^2 + y\sec^2(yz)] = -2xz$$

$$\frac{\partial z}{\partial x} = -\frac{2xz}{x^2 + y\sec^2(yz)}$$ (same)