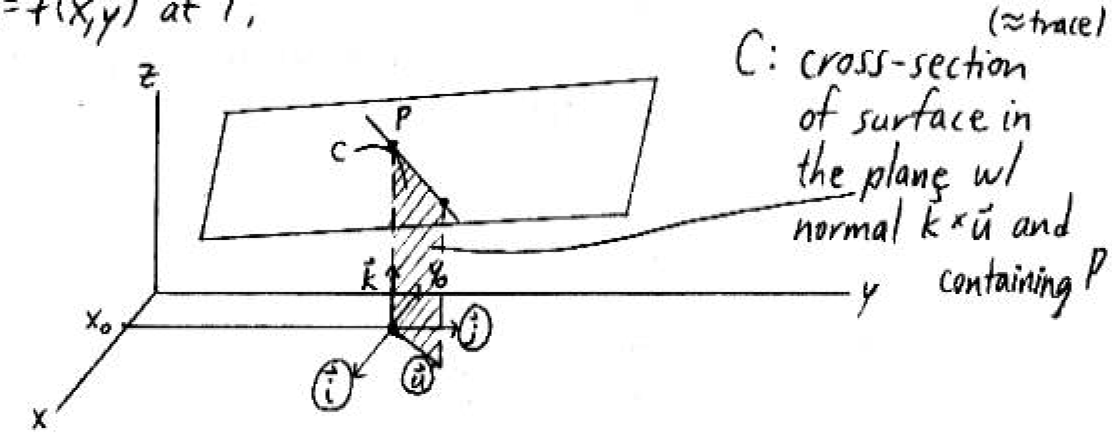
16.6: DIRECTIONAL DERIVATIVES (DDs)

Ⓐ Intro

Consider the tangent plane to the graph of $z=f(x,y)$ at P.



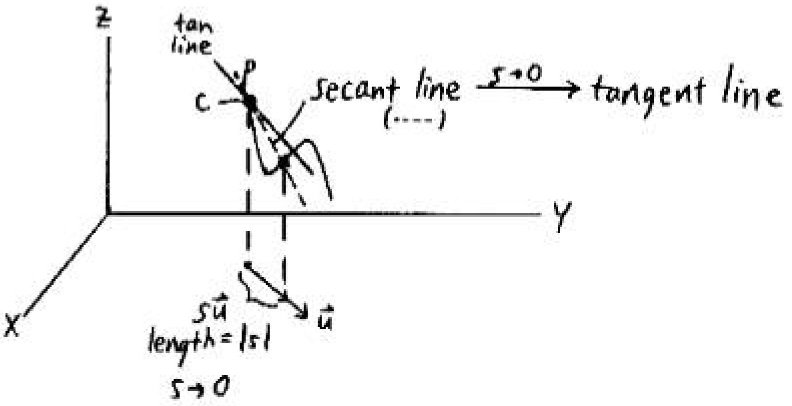
$f_x(x_0, y_0)$ = DD of f at P in the direction of $\vec{i}$

$f_y(x_0, y_0)$ = " , $\vec{j}$

$D_{\vec{u}} f(x_0, y_0)$ = " , $\vec{u}$

= slope along tangent plane at P in the direction of $\vec{u} = \langle u_1, u_2 \rangle$, a unit vector indicating "compass direction"

$$\overset{\text{Def'n}}{=} \lim_{s \to 0} \frac{f(x_0 + su_1, y_0 + su_2) - f(x_0, y_0)}{s}$$



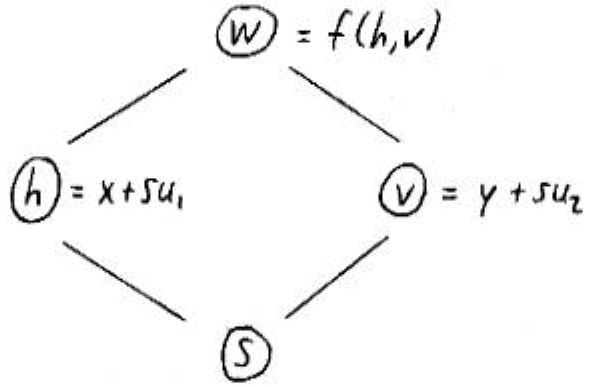
Not weighted average of f_x, f_y, since $u_1+u_2 \neq 1$ ($=1$ if $\vec{i}, \vec{j}$)

$$\boxed{\begin{aligned} D_{\vec{u}} f(x,y) &= f_x(x,y)\,u_1 + f_y(x,y)\,u_2 \\ &= \langle f_x(x,y), f_y(x,y)\rangle \cdot \langle u_1, u_2\rangle \\ &= \vec{\nabla} f(x,y) \cdot \vec{u} \\ \text{where } \vec{\nabla} f(x,y) &= \text{the } \underline{\text{gradient}} \text{ of } f \\ &= \langle f_x(x,y), f_y(x,y)\rangle \end{aligned}}$$

del, like ∂

$\vec{\nabla}$: del, the vector differential operator $\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}\rangle$

<u>Sketch of Proof</u> of ⊛



$$\frac{dw}{ds} = \underbrace{\frac{\partial w}{\partial h}}_{\substack{=f_h \\ =f_x \\ \text{at } s=0}} \underbrace{\frac{dh}{ds}}_{=u_1} + \underbrace{\frac{\partial w}{\partial v}}_{\substack{=f_v \\ =f_y \\ \text{at } s=0}} \underbrace{\frac{dv}{ds}}_{=u_2}$$

Ex $f(x,y) = 2x^2 + y^2$

ⓐ Find $\vec{\nabla} f(2,3)$.

$$\vec{\nabla} f(x,y) = \langle f_x(x,y), f_y(x,y) \rangle$$
$$= \langle 4x, 2y \rangle$$
$$\vec{\nabla} f(2,3) = \langle 4(2), 2(3) \rangle$$
$$= \boxed{\langle 8, 6 \rangle}$$

ⓑ Find the DD of f at $(2,3)$ in the direction of $\vec{a} = \langle -3, 1 \rangle$.

Find $\vec{u}$, the unit vector in the direction of $\vec{a}$.

$$\vec{u} = \frac{\vec{a}}{\|\vec{a}\|} = \frac{\langle -3, 1 \rangle}{\sqrt{(-3)^2 + (1)^2}}$$
$$= \frac{1}{\sqrt{10}} \langle -3, 1 \rangle$$

$$D_{\vec{u}}(2,3) = \vec{\nabla} f(2,3) \cdot \vec{u}$$
$$= \langle 8, 6 \rangle \cdot \frac{1}{\sqrt{10}} \langle -3, 1 \rangle$$
$$= \frac{1}{\sqrt{10}} (-24 + 6)$$
$$= -\frac{18}{\sqrt{10}}$$
$$\approx -5.7$$

ⓒ Find the DD of f at (2,3) in the direction of $\underbrace{\langle -3, 4 \rangle}_{\text{Normalize}}$.

$\downarrow$

$\vec{u}$ (a new $\vec{u}$)

$$D_{\vec{u}}(2,3) = \vec{\nabla} f(2,3) \cdot \vec{u}$$

$$= \langle 8, 6 \rangle \cdot \frac{\langle -3, 4 \rangle}{\|\langle -3, 4 \rangle\|}$$

$$= 0$$

When do 2 vectors have a "·" of 0, geom. speaking!

<u>Note</u> $\vec{\nabla} f(2,3) \perp \langle -3, 4 \rangle$

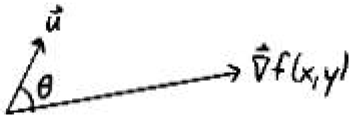
DD=? DD=0

Ⓑ Comparing DDs

The DD of f at (x,y) is maximized in the direction of $\vec{\nabla} f(x,y)$, (steepest climb) the direction of fastest increase of f. The corresponding DD $= \|\vec{\nabla} f(x,y)\|$.

| minimized
| $-\vec{\nabla} f(x,y)$ (steepest fall)
| decrease
| $-\|\vec{\nabla} f(x,y)\|$

Why?



$\cos(-\theta) = \cos\theta$

$$D_{\vec{u}} f(x,y) = \vec{\nabla} f(x,y) \cdot \vec{u}$$

$$= \|\vec{\nabla} f(x,y)\| \, \cancel{\|\vec{u}\|}^{1} \underbrace{\cos\theta}$$

Max: $=1$ $(\theta = 0)$

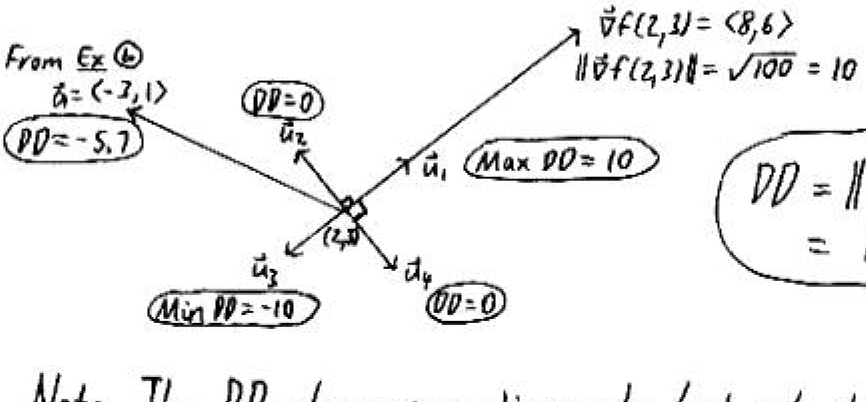
Min: $=-1$ $(\theta = \pi)$

Note: $=0$ $(\theta = \frac{\pi}{2})$

if $0 \le \theta \le \pi$

Old Ex $f(x,y) = 2x^2 + y^2$

(Not to scale) $\rightarrow$



Are you surprised it's this negative? $\rightarrow$

$$\boxed{\begin{aligned} DD &= \|\vec{\nabla} f(x,y)\| \cos\theta \\ &= 10\cos\theta \text{ here} \end{aligned}}$$

Note The DD changes continuously but not steadily wrt θ.

fastest near $\theta = \frac{\pi}{2}$ $(\vec{u}_2, \vec{u}_4)$

slowest near $\theta = 0, \pi$ $(\vec{u}_1, \vec{u}\,)$

Why? $D_\theta(DD) = -\|\vec{\nabla} f(x,y)\| \sin\theta$ $\quad \approx 0$ most extreme

DD=0

Level curve (LC) of $f(x,y) = 2x^2 + y^2$ through $(2,3)$:

Find k

$$k = f(2,3)$$
$$= 2(2)^2 + (3)^2$$
$$= 17$$

$\frac{x^2}{\frac{17}{2}} + \frac{y^2}{17} = 1$

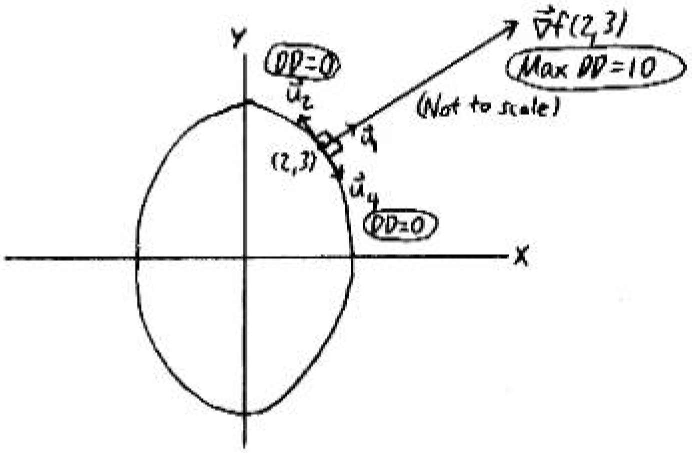
$\downarrow \quad \downarrow$

$b \approx 2.9 \quad a \approx 4.1$

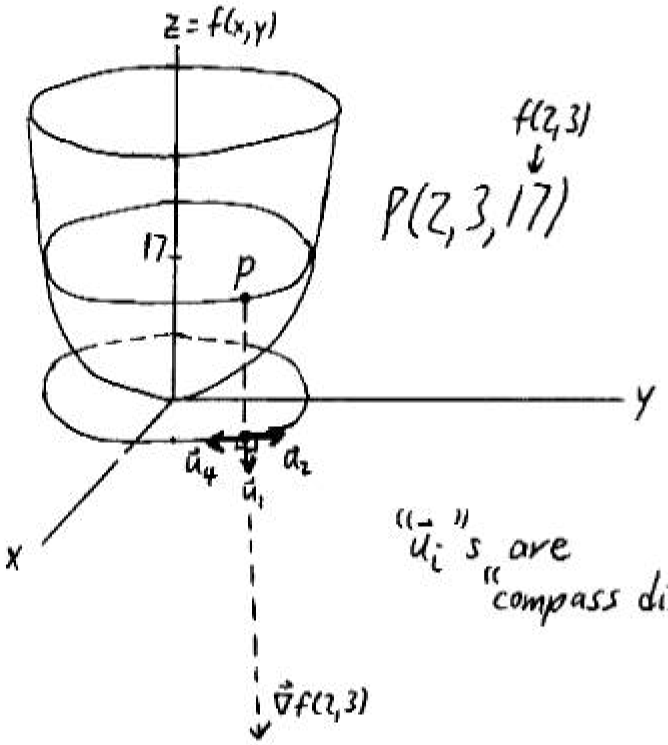
LC: $2x^2 + y^2 = 17$

At $(2,3)$, in which directions will DD=0?

$f(x,y) = 17$, a constant, for all (x,y) on LC.



DD=0 at $(2,3)$ in "tangent directions" to the LC through $(2,3)$.



$P(2,3,\underset{\uparrow}{\overset{f(2,3)}{17}})$

"$\vec{u}_i$"s are "compass directions"

<u>Path of steepest ascent</u> along the surface:

Strategy? Keep going in the direction of $\underbrace{\vec{\nabla} f(x,y)}_{\text{may change as you move}}$ on your compass.

Hard fall " " <u>descent</u> " "
" " $-\vec{\nabla} f(x,y)$ " "

<u>Ⓒ $w = f(x,y,z)$</u>

$\vec{\nabla} f = \langle f_x, f_y, f_z \rangle$

$D_{\vec{u}} f = \vec{\nabla} f \cdot \underbrace{\vec{u}}_{\text{unit } \langle u_1, u_2, u_3 \rangle}$

Level curves → Level surfaces (16.7)