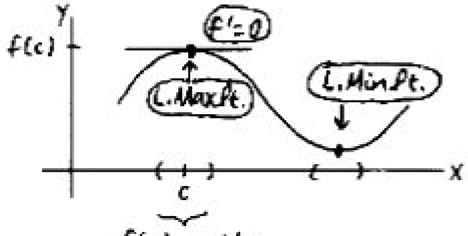
16.8: OPTIMIZATION I

Ⓐ Local/Relative Extrema of $f(x,y)$

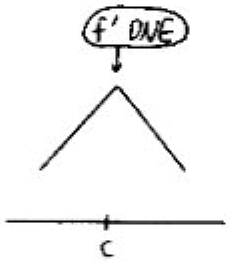
L. — Max, Min

not closed (we exclude boundary pts.)

Calc I: $y=f(x)$

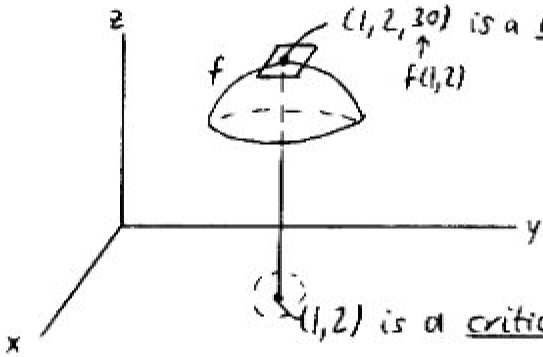


f(c) is the max value of f on some open interval containing c, a critical #



A critical # is a # in Dom(f) where $f'=0$ or DNE.
These are the only #s (candidates) where L.Max./Min. may occur. (Not "must": $\nearrow$)

Now: $z=f(x,y)$

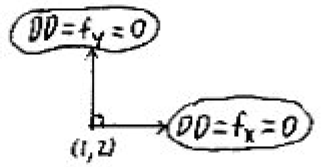


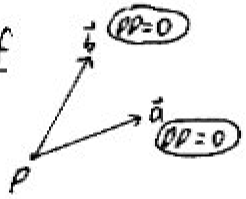
(1,2,30) is a L.Max.Pt. (max on some open disk containing (1,2))
↑
f(1,2)

(1,2) is a critical point (CP).

f has a L.Max. of 30 at (1,2).

Note 1: There is a horizontal tangent plane at $(1,2,30)$.

$\Rightarrow$ 

(If 

, $\vec{a} \nparallel \vec{b}$,
f is diff'e at P $\Bigg\} \Rightarrow$ All DDs at P are 0.)

Note 2:

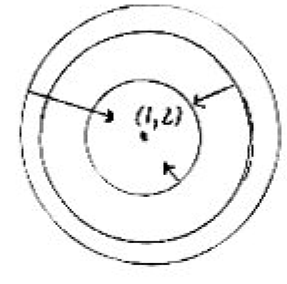
$$\vec{\nabla} f = \langle f_x, f_y \rangle$$
$$\vec{\nabla} f(1,2) = \langle 0,0 \rangle = \vec{0}$$

Why does this make sense?

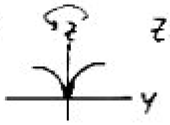
If you're at the North Pole, can you go further North?

If $\vec{\nabla} f \neq \vec{0}$, then f can increase in that direction,
and decrease in the opposite direction.
We can't be at an L. Max./Min. Pt.

Level Curves of f



$\vec{\nabla} f$ shrinks as we move towards $(1,2)$.

Note 3: $f(x,y) = \sqrt[3]{x^2+y^2}$ from: 

$z = \sqrt[3]{y^2}$

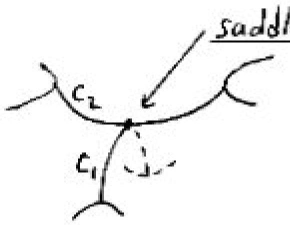

L. Min. Pt.
f_x, f_y DNE
(In fact, all DDs at $(0,0)$ DNE.)

Def'n (a,b) is a CP $\iff$
⓪ (a,b) is in $Dom(f)$
① $f_x(a,b)=0$ and $f_y(a,b)=0$ (i.e., $\vec{\nabla} f(a,b)=\vec{0}$)
or ② either DNE (i.e., $\vec{\nabla} f(a,b)$ DNE)

CPs are the only places where L. Max./Min. Pts. can occur.

Ex 2 (p.863) CP where neither occurs

$$f(x,y)=y^2-x^2 \quad \text{(Hyp. paraboloid)}$$



Discriminants help us classify. b^2-4ac helped us classify roots of a quadratic func. as real or imaginary.

Ⓑ Classifying CPs

Assume the 2^{nd} PDs of f are cont. where we care
$\Rightarrow f_{xy}=f_{yx}$

Do we care if we switch f_{xy}, f_{yx}?

The discriminant of f = "D" or "$D(x,y)$"
$= \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}$
$= f_{xx}f_{yy}-(f_{xy})^2$

real, sym. matrix $\Rightarrow$ all evals real

Larson 891 (6ed) - can find CPs in higher dims, but test is ugly

2^{nd} Derivative Test for $f(x,y)$

At a CP (a,b) where $\vec{\nabla} f = \vec{0}$, not DNE,

*① If $D > 0$,

ⓐ if $f_{xx} < 0 \Rightarrow$ concave down ☹ (word association works) fails
$\Rightarrow$ L.Max. at (a,b)

from How to Ace

ⓑ $>$ up ☺
L.Min.

** ② If $D < 0 \Longrightarrow$ saddle pt. (SP)

③ If $D = 0 \Longrightarrow$ no info

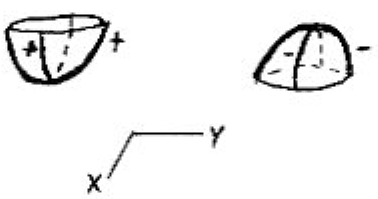
In Calc I, SDT, what did $f'' = 0$ tell us?

* For ①, you can use f_{yy}.

If $f_{xx}, f_{yy} > 0$ and f_{xy} not too influential $\Rightarrow D > 0$

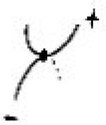
If $D = \underbrace{f_{xx} f_{yy}}_{>0} - (f_{xy})^2 > 0$, then

$\Rightarrow f_{xx}, f_{yy}$ have same sign



(x-,y-concavities both up or both down and f_{xy} not too influential)

** ② Ex $f_{xx} < 0,\ f_{yy} > 0 \Rightarrow D < 0 \Rightarrow SP$



© Exs

M121 #18
3rd ed.

<u>Ex</u> Find the local extrema and saddle points of $f(x,y) = -x^2 - y^3 - 6x + 3y + 4$.

<u>Step 1</u>: Find CPs.

Technically, $f_x(x,y)$

$$\left.\begin{aligned} f_x &= \underbrace{-2x-6}_{\text{never DNE}} \overset{\text{set}}{=} 0 \\ f_y &= \underbrace{-3y^2+3}_{\text{never DNE}} \overset{\text{set}}{=} 0 \end{aligned}\right\} \text{Solve system.}$$

$-2x-6=0$ <u>and</u> $-3y^2+3=0$

$x=-3$ $\qquad y^2=1$

$y=\pm 1$

CPs: $(-3,1)$
$(-3,-1)$

<u>Note</u>: If we had... ⍟

$$\begin{cases} x+y=0 \Leftrightarrow y=-x \\ x-y^2=0 \end{cases}$$

$$\Leftrightarrow \begin{cases} y=-x \\ x-(-x)^2=0 \end{cases}$$

$x-x^2=0$

$x(1-x)=0$

$x=0 \Rightarrow y=0$

or ⍟

$x=1 \Rightarrow y=-1$

CPs: $(0,0)$
$(1,-1)$

<u>Step 2</u>: Find f_{xx}, D.

$f_x = -2x-6$

$f_{xx} = -2$

$f_y = -3y^2+3$

$f_{yy} = -6y$

$f_{xy} = f_{yx} = 0$

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} -2 & 0 \\ 0 & -6y \end{vmatrix} = 12y$$

Step 3: Classify CPs.

CP	$D = 12y$	$f_{xx} = -2$	Conclusion
$(-3, 1)$	$12(1) = 12 > 0$	-2 ☹	L. Max.
$(-3, -1)$	$12(-1) = -12 < 0$	(irrelevant)	SP

Step 4: Find f values at CPs.

L. Max. Pt. at $(-3, 1, f(-3,1))$
$(-3, 1, 15)$ ← L. Max. Value

SP at $(-3, -1, f(-3,-1))$
$(-3, -1, 11)$

Larson #59
$x^4 - 2x^2 + y^2$
has 2 rel. min.
but no rel. max.

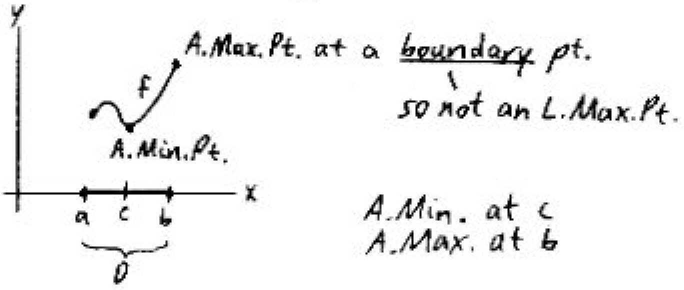
ⓓ What if the domain, D, is closed? (Not on tests)

Calc I

Extreme Value Thm. (EVT)

If f is cont. on $[a,b]$, a closed interval
$\Rightarrow$ There exist absolute max. and min. in $[a,b]$.
A.
(or global)

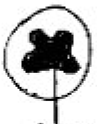
Ex



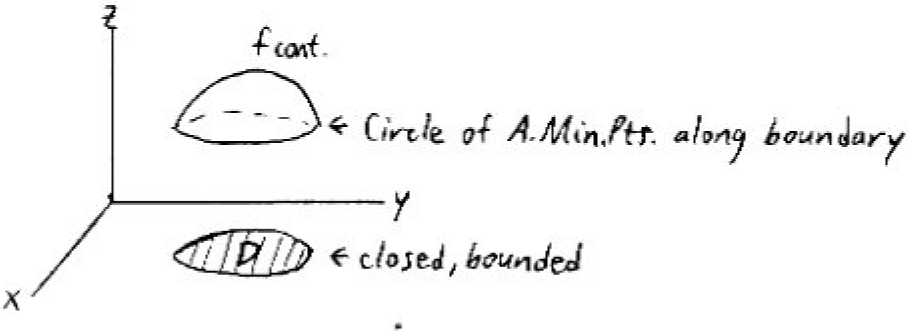
A.Min. at c
A.Max. at b

Now

Unbounded $\not\Rightarrow$ Bounded = is a subregion of some disk



trapped by circle



If D is open $\Longrightarrow$ no boundary extrema to worry about

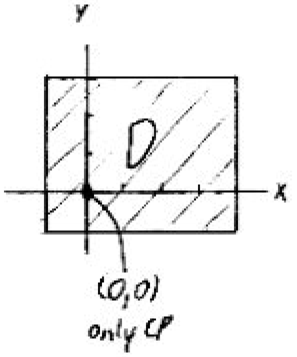
If D is closed $\Longrightarrow$ examine boundary for possible A.Min./Max. Pts.

EVT Extension: If f is cont. on a closed D
$\Rightarrow$ There exist A.Max. and A.Min. in D.

Ex (#25) Find absolute extrema of $f(x,y) = x^2 + 2xy + 3y^2$
on $D = \{(x,y) \mid -2 \le x \le 4 \text{ and } -1 \le y \le 3\}$

Collect candidates where A.Max./Min. Pts. might appear.

① Find CPs in D (excluding boundary).

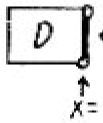


② Examine the 4 sides.

closed interval ∃ redundant work; see ①

Split or Param w/t

Ex

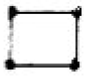


$$f(4,y) = (4)^2 + 2(4)y + 3y^2$$
$$= \underbrace{16 + 8y + 3y^2}_{g(y)}$$

Calc I: $g'(y) = 0 \Rightarrow y = -\frac{4}{3}$

<u>but</u> $(4, -\frac{4}{3})$ is not in D, so toss it!

③ Find corners of D.



Beauty Contest

④ Compare the f values at <u>all</u> our candidates.

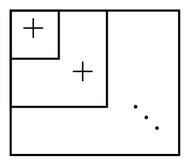
Highest f value $\Rightarrow$ A.Max. value on D; Lowest $\Rightarrow$ A.Min.

PART E: FOOTNOTES

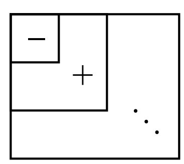
Extending the 2nd Derivative Test

If you have a nice function of *n* variables, you will construct an *n* x *n* real symmetric matrix consisting of *n*th-order partial derivatives; such a matrix only has real eigenvalues. When classifying a critical point (CP), we consider the signs of the determinants of all the upper left square submatrices (1 x 1, 2 x 2, etc.).

• If they are all positive, the matrix is called positive definite, and all of its eigenvalues are positive. The CP corresponds to a local min.



• If they alternate in sign from negative to positive, etc., the matrix is called negative definite, and all of its eigenvalues are negative. The CP corresponds to a local max.



• If they are all nonzero, and neither of the two above configurations occur, then the CP corresponds to a saddle point (SP).

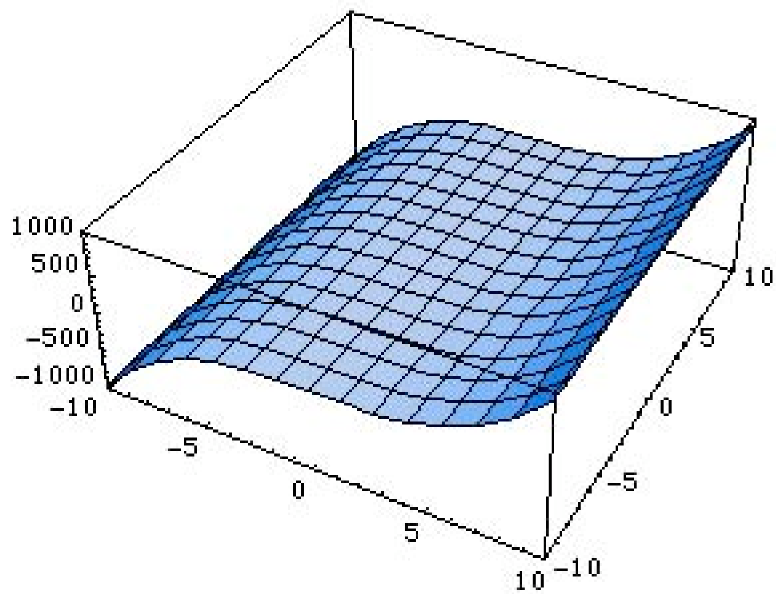
Observe that the notes on 16.8.4 are consistent with all of this.

Defining a Saddle Point (SP)

The Harper Collins Dictionary of Mathematics:
"A point on a surface that is a maximum in one planar cross-section and a minimum in another."

Visualizing a hyperbolic paraboloid helps.

The definition may vary. Are degenerate "ties" allowed along a cross-section, like for horizontal lines? Also, for example, are the points along the y-axis saddle points if we have the graph of the "snake cylinder" $f(x,y) = x^3$?



Orientation of axes: $<^{y}_{x}$

That's debatable. Using the *Harper Collins* definition, I don't believe they would be; the thing just doesn't look like a "saddle" along the y-axis. But it is true that there are higher and lower points "immediately around" those points. Incidentally, $D = 0$ everywhere for this function, so the 2nd Derivative Test says nothing.

See: http://en.wikipedia.org/wiki/Saddle_point