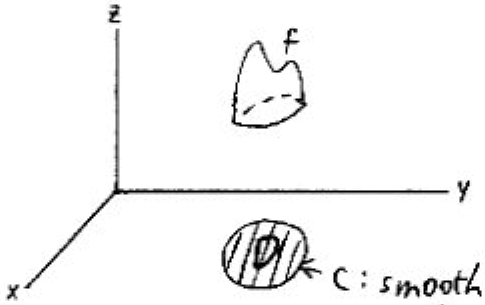
16.9: CONSTRAINED OPTIMIZATION – LAGRANGE MULTIPLIERS

(A) Intro

In 16.8 (D)

See Larson 6ed p. 906

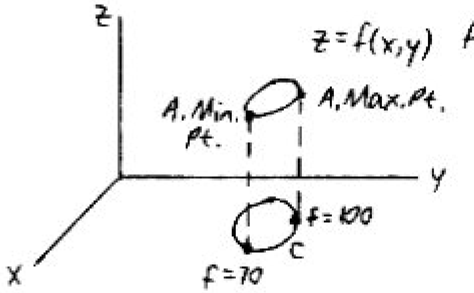


← C: smooth boundary curve with eq. $g(x,y)=0$

I'd like to analyze f along C.

Now, what if C is my [restricted] domain?

Like a stripe along the surface



"where we care": In vicinity of pts. along C? 16.4: f_x, f_y cont. in open region ⇒ f diff'e there ⇒ f cont.

Another EVT Extension

If f is cont. on a closed curve (like ○) or on a curve that includes its endpts. (like ∿)

⇒ There exist A.Max. and A.Min. along the curve.

Note: This extends to surfaces and their boundaries in higher dims.

Assume $\vec{\nabla} f$ is cont., $\vec{\nabla} g \neq \vec{0}$ where we care.

Goal: Find L. or A. Max./Min. of $f(x,y)$ subject to the constraint $g(x,y)=0$.
(s.t.)

Calc I: Pigpen problems!
100 ft. of fencing
Maximize Area $f(x,y)=xy$
s.t. $2x+2y=100$
(i.e., $\underbrace{2x+2y-100}_{g(x,y)}=0$)
$\Rightarrow y=50-x$

Note 1 If you can solve $g(x,y)=0$ for y in terms of x or x in terms of y

If can solve for y in terms of x $\Rightarrow$ let $t=x$, or $x=t$.

or If you can parameterize x and y in terms of t, then you can do it, sub into $f(x,y)$, and use Calc I.

$x \quad y$ → t

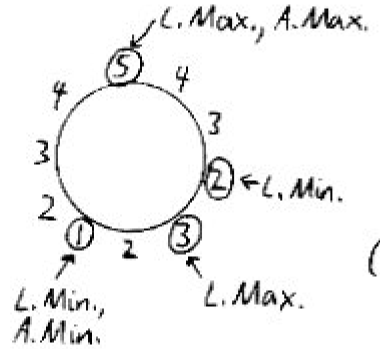
$x=\cos t$
$y=\sin t$

Note 2

Ex (Not the "f" from 16.9.1.)

Labels are f values:

To trace the corresp. graph of f, move your finger so it has height z or f.



(OK, maybe the L. Max. value is 3.1, not 3. Shut up! :))

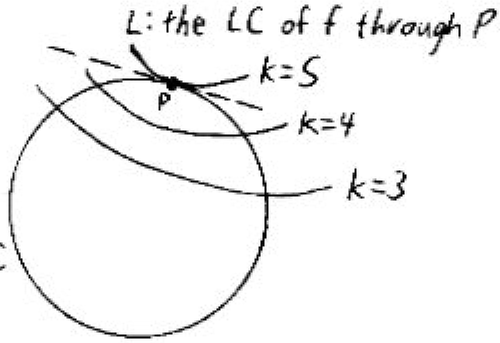
If C is a closed curve like or is like , then A. Max./Min. are L. Max./Min.
(We're assuming f is cont. where we care.)

If P C Q, then check for possible A. Max./Min. at P, Q. Can't have L. Max./Min. there.

(B) Lagrange's Thm.

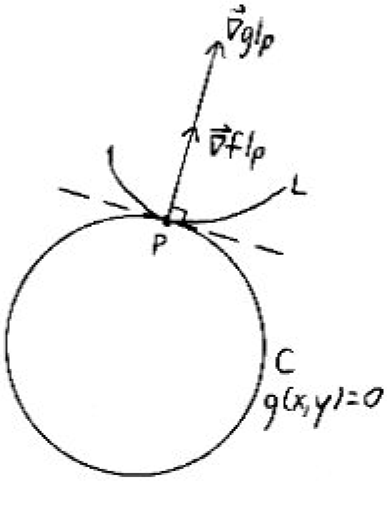
Idea Consider the level curves (LCs) of f.

Where are the local extrema of f on C?



Where the LCs of f barely touch C. (Why? See 16.9.11)

At P, L and C share the same tangent line.



L is a LC of f,
C " " g

$\Longrightarrow \vec{\nabla} f \parallel \vec{\nabla} g$ at P

<u>Lagrange's Thm.</u>

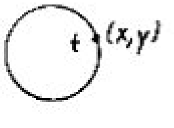
<u>If</u> P is a L. Max./Min. locale along C,
<u>then</u> there is a real #, λ (lambda), such that

$$\vec{\nabla} f = \lambda \vec{\nabla} g \text{ at } P.$$

λ is a <u>Lagrange multiplier</u>.

<u>Proof</u> (Optional)

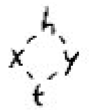
Let $\vec{r}(t) = \langle x(t), y(t) \rangle$ be a smooth param. of C.
Let $h(t) = f(x(t), y(t))$.



$h(t) = f(x,y)$

$h'(t)$ or $\frac{dh}{dt} = 0$ at P, a L. Max./Min. locale along C.
(can't be "DNE" where $\vec{\nabla} f$ cont.)

By Chain Rule,



At P,

$$\frac{dh}{dt} = \underbrace{\frac{dh}{dx}}_{= f_x} \frac{dx}{dt} + \underbrace{\frac{dh}{dy}}_{= f_y} \frac{dy}{dt} = 0$$

$$\underbrace{\langle f_x, f_y \rangle}_{\vec{\nabla} f} \cdot \underbrace{\left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle}_{\vec{r}\,'} = 0$$

$\vec{r}$ is tangent to C at P
C is a LC of g
$\Rightarrow$ $\vec{\nabla} g \perp \vec{r}\,'$

$\vec{\nabla} f \perp \vec{r}\,'$ and $\vec{\nabla} g \perp \vec{r}\,'$ $\Rightarrow$ $\vec{\nabla} f \parallel \vec{\nabla} g$

$\vec{\nabla} f = \lambda \vec{\nabla} g$
for some real λ

Ⓒ Method

<u>Goal</u>: Find L. or A. Max./Min. of $f(x,y)$
subject to $g(x,y)=0$.

To find the candidates (x,y) for L. Max./Min. locales,

solve $\begin{cases} \vec{\nabla} f(x,y) = \lambda \vec{\nabla} g(x,y) \\ g(x,y) = 0 \end{cases}$ ← ensures (x,y) is on C

for $(x,y,\underset{\uparrow}{\lambda})$

can differ for different candidates (x,y);
don't have to find (λ: means to end)

If you're looking for A. Max./Min.,
examine any endpoints of C.

Method extends to higher dimensions. (LCs → LSs)

<u>Note</u> If there are 2 constraints, $g(x_1, \ldots, x_n)=0$
and $h\underbrace{(x_1, \ldots, x_n)}_{\text{I'll omit}}=0$,

solve $\begin{cases} \vec{\nabla} f = \lambda \vec{\nabla} g + \mu \vec{\nabla} h \\ g = 0 \\ h = 0 \end{cases}$ ← linear combo of $\vec{\nabla} g, \vec{\nabla} h$

Intersection of $g=0, h=0$

for (x, y, z, λ, μ)

In Ex. 4 on pp. 879-80, can param. this, use Calc I.

Stewart 12.8, #3
Math 20C
not in ET, Se

① Ex Find the L. Max./Min. of $f(x,y) = xy$ subject to $9x^2 + y^2 = 4$.

Sol'n

Not really necessary here, but good form. We then use "$g = 0$"

$$9x^2 + y^2 = 4$$
$$\underbrace{9x^2 + y^2 - 4}_{g(x,y)} = 0 \quad \text{(Isolate 0.)}$$

Solve $\begin{cases} \vec{\nabla} f(x,y) = \lambda \vec{\nabla} g(x,y) \\ g(x,y) = 0 \quad \leftarrow ⊛ \end{cases}$

$$\vec{\nabla} f(x,y) = \lambda \vec{\nabla} g(x,y)$$
$$\langle f_x, f_y \rangle = \lambda \langle g_x, g_y \rangle$$
$$\langle y, x \rangle = \lambda \langle 18x, 2y \rangle$$

Solve $\begin{cases} ① \quad y = \lambda(18x) \\ ② \quad x = \lambda(2y) \\ ⊛ \quad 9x^2 + y^2 - 4 = 0 \end{cases}$

① $y = \lambda(18x) \Rightarrow \lambda = \frac{y}{18x}$ (if $x \neq 0$) or $x = 0 \overset{y=\lambda(18x)}{\Downarrow} y = 0$

② $x = \lambda(2y) \Rightarrow \lambda = \frac{x}{2y}$ (if $y \neq 0$) or $y = 0 \overset{x=\lambda(2y)}{\Downarrow} x = 0$

Like
$t = \ldots$
for param
eqs. for line.

① and ② $\Rightarrow \underbrace{\lambda = \frac{y}{18x} = \frac{x}{2y}}$ or $(x=0, y=0)$

$\Rightarrow 2y^2 = 18x^2$

$y^2 = 9x^2$

Use ④

If $y^2 = 9x^2$,

$9x^2 + y^2 - 4 = 0$
$9x^2 + 9x^2 - 4 = 0$
$18x^2 = 4$
$x^2 = \frac{2}{9}$
$x = \pm \frac{\sqrt{2}}{3}$

For both $x = \pm \frac{\sqrt{2}}{3}$

$\Rightarrow x^2 = \frac{2}{9}$

$y^2 = 9x^2$
$y^2 = 9\left(\frac{2}{9}\right)$
$y^2 = 2$
$y = \pm\sqrt{2}$

If $(x=0, y=0)$,

$0 + 0 - 4 = 0$ NO!

(0,0) does <u>not</u> lie on C.
Toss it!

4 candidates: $\left(\pm \frac{\sqrt{2}}{3}, \pm\sqrt{2}\right)$

All 4 combos
Sign free-for-all!

"$\pm$"s can be ambiguous.

<u>Note</u>: If we had $x = \pm 2$, $y = 3x$
$x = 2 \Rightarrow y = 6$
$x = -2 \Rightarrow y = -6$

Only 2 cands.: $(2,6)$
$(-2,-6)$

Evaluate f at the candidates.

$f(x,y) = xy$

$f(\underbrace{\tfrac{\sqrt{2}}{3}, \sqrt{2}}_{A}) = (\tfrac{\sqrt{2}}{3})\sqrt{2}) = \tfrac{2}{3}$ L., A. Max. value

$f(\underbrace{\tfrac{\sqrt{2}}{3}, -\sqrt{2}}_{B}) = -\tfrac{2}{3}$ L., A. Min. value

$f(\underbrace{-\tfrac{\sqrt{2}}{3}, \sqrt{2}}_{\hat{C}}) = -\tfrac{2}{3}$ L., A. Min. value

("C" is our ellipse)

$f(\underbrace{-\tfrac{\sqrt{2}}{3}, -\sqrt{2}}_{D}) = \tfrac{2}{3}$ L., A. Max value

Why?

Ellipse

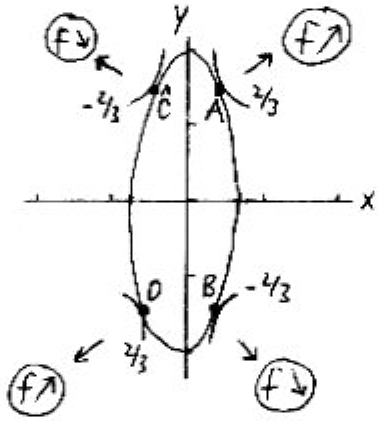
$9x^2+y^2=4$

$\frac{x^2}{\frac{4}{9}} + \frac{y^2}{4} = 1$

$a=2$

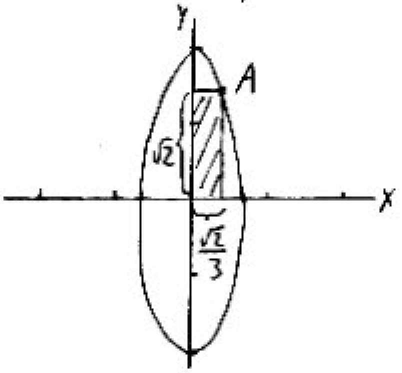
$b=\frac{2}{3}$

Analyze LCs of f:



Application

Find the dimensions of the rectangle of max. area in QI whose corners are on (0,0), the x- and y-axes, and the ellipse $9x^2+y^2=4$.



$\frac{\sqrt{2}}{3}$ units by $\sqrt{2}$ units

Area $= \frac{2}{3}$ units^2

(E) Strategies for Classifying Candidates

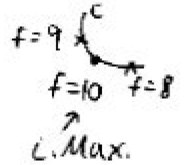
The test is ugly!

(1) Consider the graph (or LCs) of f.

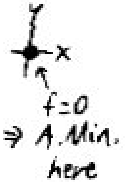
(2) Compare the f values of the candidates.

(3) If there's only 1 candidate,
try to trace the behavior of f near it along C, maybe by examining other pts. on C.

2 pts. just in case a SP?



(4) Examine algebraic properties of f, especially with respect to range. Ex $f(x,y) = x^2 + y^2$

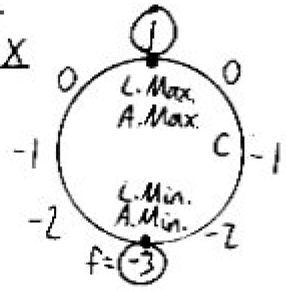


I said this in 16.4.5, too.

(5) If you're looking for A. Max, Min., examine any endpoints of C.

(6) If f is cont. on a closed curve C (like ○c), and there are only two candidates for local extrema, then one must be a L.Max. (and an A.Max.) and the other must be a L.Min. (and an A.Min.).

Ex



Sign analyses may simplify classification! -3 vs. $+1$

By an EVT, these exist. They must occur at L.Max./Min. if C is closed. ○c

16.9.10

(E) Strategies for Solving Systems

(1) Eliminate variables one-by-one.

(2) Solve for λ in the $\vec{\nabla} f = \lambda \vec{\nabla} g$ eqs. and equate.

(3) Solve for x, y in terms of λ. $\Rightarrow$ Get eq. in λ.

(4) Multiply both sides of an eq. by something to make elimination easier. (Beware of cases where this something is 0.)

Be mindful of this when canceling! Factoring is preferable.

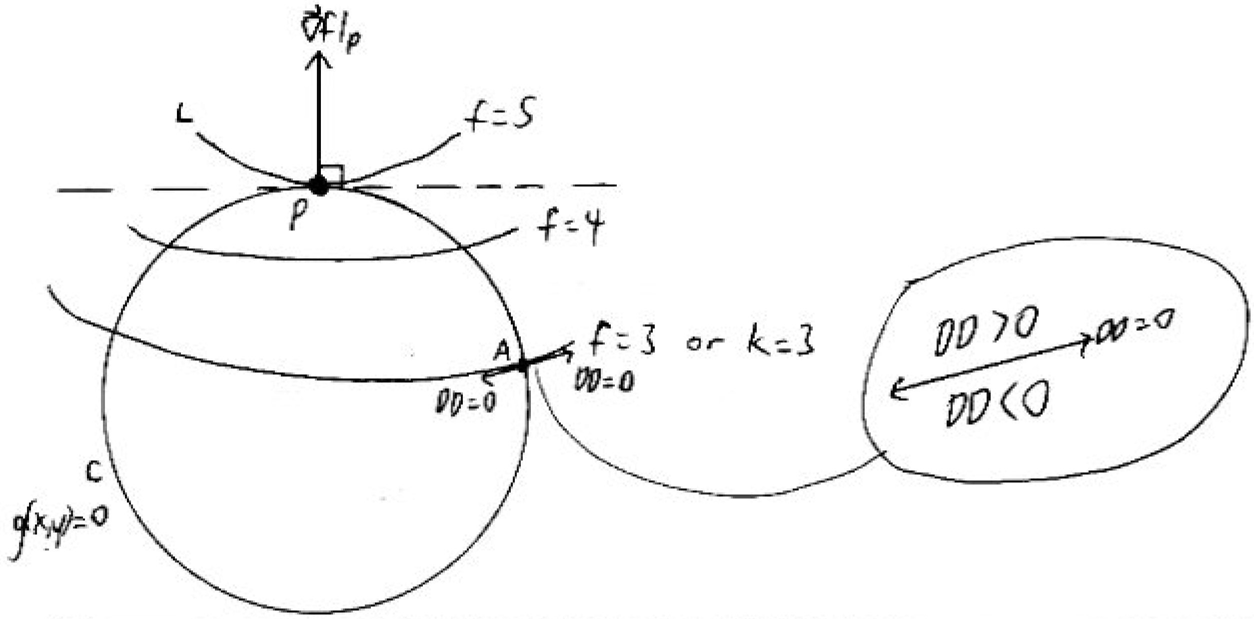
How to Ace Rest p.167

(6) Footnotes

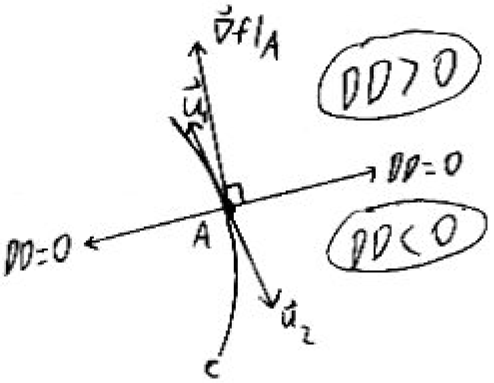
a level curve of g

Why focus on where L, C are tangent? (16.9.3)

a level curve of f



Zoom in: on A



From A, we have 3 choices to try to max or min f:

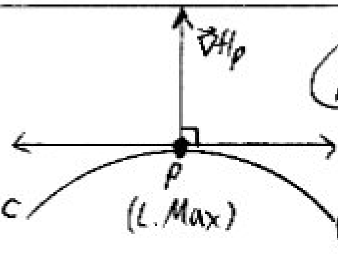
1. Stay at A.
2. ↑ $\vec{u}_1$
3. ↓ $\vec{u}_2$

A can't be a L. Max or Min.

To ↗ f, go in the direction of $\vec{u}_1$.
    ↘ $\vec{u}_2$.

If you're at the North Pole, can you go further North?

Zoom in: on P



DD>0 but we'd go off C!

DD<0

DD of f at a point P in the direction of $\vec{u}$ (unit tangent at P to C)

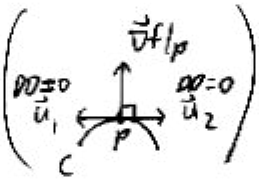
$= D_{\vec{u}} f(P)$

$= (\vec{\nabla} f|_P) \cdot \vec{u}$

$= \dfrac{(\vec{\nabla} f|_P) \cdot \vec{u}}{\|\vec{u}\|}$ $\leftarrow = 1$

$= comp_{\vec{u}} (\vec{\nabla} f|_P)$

(This $= 0$) $\Leftrightarrow$ $\underbrace{(\vec{\nabla} f|_P)}_{\perp \text{ to } C \text{ at } P} \perp \underbrace{\vec{u}}_{\text{tangent to } C}$



As in Calc I, we care where DD = 0.

<u>Note</u>: If $\vec{\nabla} f|_P = \vec{0}$, we have DD = 0 automatically. In 16.8, we knew that pts. where $\vec{\nabla} f = \vec{0}$ were interesting, anyway.