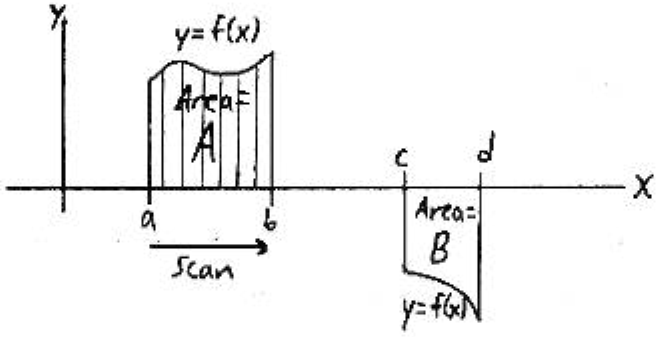
CH. 17: MULTIPLE INTEGRALS

17.1: DOUBLE INTEGRALS ($\iint$), and 17.2: AREA and VOLUME

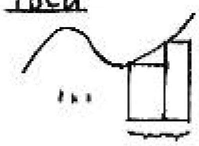
Ⓐ Intro

Calc I



$$\boxed{\int_a^b f(x)\,dx = A} \qquad \boxed{\int_c^d f(x)\,dx = -B}$$

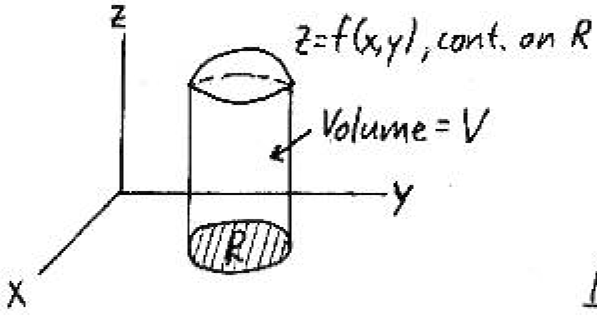
Idea



Add areas

Widest width → 0
(⇒ # rectangles → ∞)

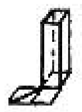
Now



$$\boxed{\iint_R f(x,y)\,dA = V}$$

Below xy-plane ⇒ −V

Idea

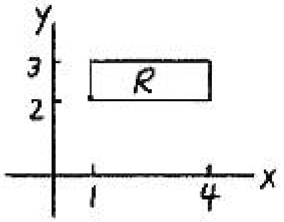


Add volumes

Longest diagonal → 0
(⇒ # boxes → ∞)

(B) Rectangular R

<u>Ex</u> R is given by $\{(x,y) \mid 1 \le x \le 4 \text{ and } 2 \le y \le 3\}$.



Find the volume between the graphs of
$\underbrace{z}_{f(x,y)} = x^2y + y^2$ and $\underbrace{z=0}_{xy\text{-plane}}$ over R.

<u>Sol'n</u>

<u>Note</u> $x^2y + y^2 \ge 0$ on R, so our solid lies entirely $\underset{\text{(and on)}}{\text{above}}$ the xy-plane.

Guido Fubini (1879- 1943)

By Fubini's Theorem,

$$\iint_R f(x,y)\, dA =$$

(#1) $$\int_2^3 \int_1^4 \underbrace{(x^2y + y^2)}\, dx\, dy$$

Easier to do partial ∫ wrt x, so do it 1st

Then, ∫ wrt y

Inside-out

<u>or</u> (#2) $$\underbrace{\int_1^4 \underbrace{\int_2^3 (x^2y + y^2)\, dy}\, dx}$$

(#1) $\int_2^3 \left[\int_1^4 (x^2 \underbrace{y} + \underbrace{y^2})\,dx \right] dy$

Treat y as a constant.

$= \int_2^3 \left[\left(\frac{x^3}{3}\right) y + y^2 x \right]_{x=1}^{x=4} dy$

↑ Optional reminder

Think: $\int x^2 \cdot 7\,dx = \left(\frac{x^3}{3}\right) \cdot 7 + C$

$\int 7\,dx = 7x + C$

$= \int_2^3 \left(\left[\frac{(4)^3}{3} y + y^2(4) \right] - \left[\frac{(1)^3}{3} y + y^2(1) \right] \right) dy$

$= \int_2^3 \left(\frac{64}{3} y + 4y^2 - \frac{1}{3} y - y^2 \right) dy$

$= \int_2^3 (3y^2 + 21y)\,dy$ (Calc I!)

$= \left[y^3 + 21\left(\frac{y^2}{2}\right) \right]_2^3$

$= \left[(3)^3 + 21 \cdot \frac{(3)^2}{2} \right] - \left[(2)^3 + 21 \cdot \frac{(2)^2}{2} \right]$

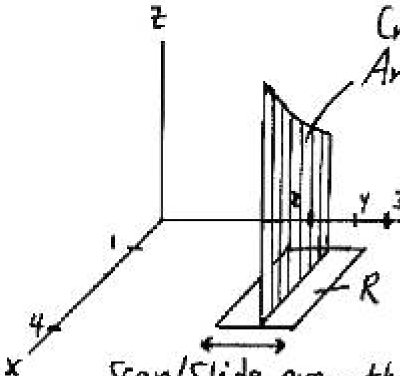
Can do 13

$= \boxed{\frac{143}{2} \text{ or } 71.5 \text{ cubic units}}$

<u>Idea</u>

Cross-Sectional Area $= A(y)$

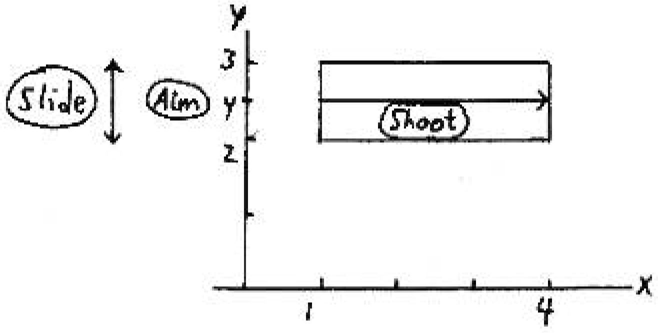
obtained by scanning line segments (S wrt x)



Scan/Slide over these areas (S)

$\int_2^3 \underbrace{\underbrace{\int_1^4 (x^2 y + y^2)\,dx}_{\substack{= A(y) \\ \text{for fixed } y \text{ bet. } 2,3}} \,dy}_{\text{Slide (S wrt y)}}$

How did we scan over R?



Aim Fix a y-value between 2, 3.
y is the outer variable.

Shoot From $x=1$ to $x=4$.
x is the inner variable.

Slide ($\int$) From $y=2$ to $y=3$.
outer
y: final scanning direction ↕

(C) Other Rs

Variation on 17.2, #32 (no "2")

Ex Find the volume, V, of the solid bounded by the graphs of:

You figure out

$y = 2x^3$
$y = x^4$
} (Determine R in xy-plane)

We'll see why

$z - x - y = 4 \Leftrightarrow z = x + y + 4$ (Top surface) (Plane)

$z = 0$ xy-plane (Bottom surface)

(1) What is R?

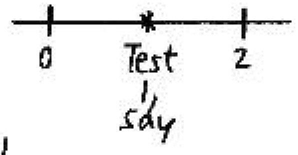
(1a) Find any Intersection Points (Just "x" coords.? "y"?)

Solve $\begin{cases} y = 2x^3 \\ y = x^4 \end{cases}$

$$\Rightarrow 2x^3 = x^4$$
$$0 = x^4 - 2x^3$$
$$0 = \underbrace{x^3}_{0}\underbrace{(x-2)}_{2}$$

$x = 0, 2$

(1b) Which graph is on top? bottom?



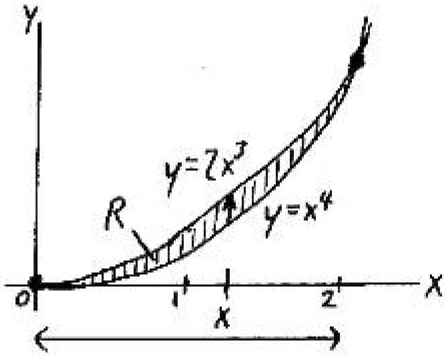
At $x=1$,

$2x^3 = 2(1)^3 = 2$ top 2 > 1, so

$x^4 = (1)^4 = 1$ bottom

where $0 < x < 2$

(1c) Sketch R (maybe - I may want this on a test)



(2) Set up $\iint$

Idea

Aim Fix x $(0 \le x \le 2)$
outer variable

Shoot $\uparrow$ From $y = x^4$ (bottom) to $y = 2x^3$ (top) y: inner variable

Slide $\leftrightarrow$ From $x = 0$ to $x = 2$
outer variable is overall "scan" variable

Surfaces

On R, $z = x + y + 4 > 0$ top

$\underbrace{x + y}_{\geq 0}$

$z = 0$ bottom

$$V = \iint_R \left[\underbrace{(x+y+4)}_{\text{top } z} - \underbrace{(0)}_{\text{bottom } z} \right] dA$$

Note It's OK if "bottom z" is sometimes negative on R, as long as it's below or at "top z."

$$= \int_{x=0}^{x=2} \int_{y=x^4}^{y=2x^3} (x+y+4)\, dy\, dx$$

must be constant relative to x, y

can depend on x but not y if $\int$ wrt y

Shoot

Slide

"$x =$"
"$y =$" } optional but help!

③ Evaluate $\iint$ (I may or may not ask for this.)

$$V = \int_{x=0}^{x=2}\left[\int_{y=x^4}^{y=2x^3} (x+y+4)\,dy\right]dx$$

$$= \int_{x=0}^{x=2}\left[xy + \tfrac{1}{2}y^2 + 4y\right]_{y=x^4}^{y=2x^3} dx$$

$$= \int_{x=0}^{x=2}\Big(\left[x(2x^3) + \tfrac{1}{2}(2x^3)^2 + 4(2x^3)\right] - \left[x(x^4) + \tfrac{1}{2}(x^4)^2 + 4(x^4)\right]\Big)dx$$

$$= \int_0^2 \left(2x^4 + \tfrac{1}{2}(4x^6) + 8x^3 - \left[x^5 + \tfrac{1}{2}x^8 + 4x^4\right]\right)dx$$

You're in danger of forgetting

$$= \int_0^2 \left(2x^4 + 2x^6 + 8x^3 - x^5 - \tfrac{1}{2}x^8 - 4x^4\right)dx$$

$$= \int_0^2 \left(-\tfrac{1}{2}x^8 + 2x^6 - x^5 - 2x^4 + 8x^3\right)dx$$

$$= \left[-\tfrac{1}{2}\left(\tfrac{x^9}{9}\right) + 2\left(\tfrac{x^7}{7}\right) - \tfrac{x^6}{6} - 2\left(\tfrac{x^5}{5}\right) + \cancel{8}^{2}\left(\tfrac{x^4}{\cancel{4}}\right)\right]_0^2$$

$$= \left[-\tfrac{1}{2}\left(\tfrac{2^9}{9}\right) + 2\left(\tfrac{2^7}{7}\right) - \tfrac{2^6}{6} - 2\left(\tfrac{2^5}{5}\right) + 2(2)^4\right] - [0]$$

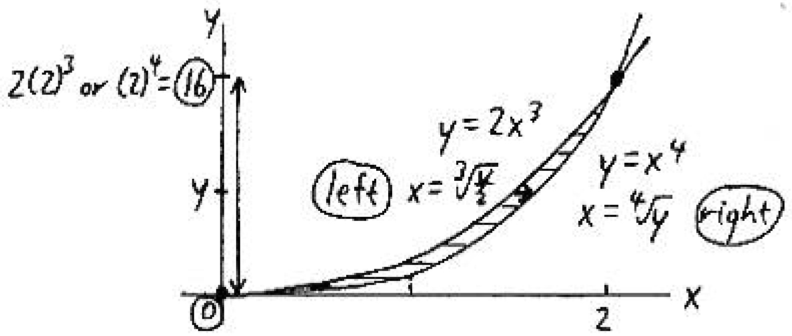
$$= \boxed{\frac{5248}{315}}$$

① Reversing the Order of Integration

Same Ex $\int_{x=0}^{x=2}\int_{y=x^4}^{y=2x^3} (x+y+4)\,dy\,dx \Rightarrow \iint_R (x+y+4)\,dx\,dy$

Don't just switch!

Sketch R



Solve for x

$y = 2x^3$ $\qquad$ $y = x^4$

$\frac{y}{2} = x^3$ $\qquad$ $x = \cancel{\pm}\sqrt[4]{y}$ ← x in [0,2]

$x = \sqrt[3]{\frac{y}{2}}$

Aim: Fix y ← outer $(0 \le y \le 16)$

Shoot → From $x = \sqrt[3]{\frac{y}{2}}$ (left) to $x = \sqrt[4]{y}$ (right)

Slide ↕ From $y=0$ to $y=16$

All but 17.2, #1

$$\boxed{\int_{y=0}^{y=16}\int_{x=\sqrt[3]{y/2}}^{x=\sqrt[4]{y}} (x+y+4)\,dx\,dy}$$

It may be easier (or necessary) to reverse the order because of R and/or the form of $\underbrace{f(x,y)}$.

Is it easier to $\int$ wrt x or $\int$ wrt y first?

R is vertically simple →  $\longleftrightarrow x$ $\iint \sim dy\,dx$

outer/ scan

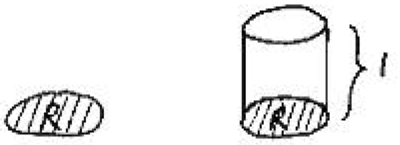
horiz. → $y \updownarrow$  $\iint \sim dx\,dy$

<u>Ex</u> (17.1.*46) $\int_0^9 \int_{\sqrt{y}}^3 \sin x^3 \, dx \, dy$

<u>BUT</u> I can't do $\int \sin x^3 \, dx$ (without series)

<u>BETTER</u> $\int \underbrace{\sin x^3}_{\text{"\#"}} \, dy = y \sin x^3$

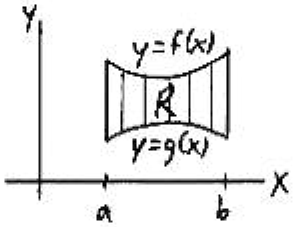
(E) Area of R



Area = Volume (numerically)

Choose $f(x,y) = 1$.

Ex



$$A = \iint_R 1 \, dA$$

$$= \int_{x=a}^{x=b} \int_{y=g(x)}^{y=f(x)} 1 \, dy \, dx$$

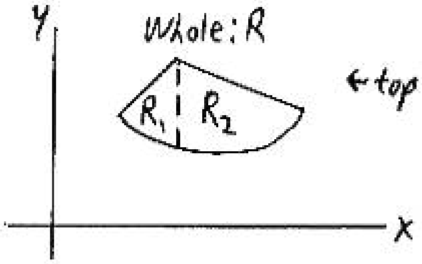
$$= \int_{x=a}^{x=b} \left[y \right]_{y=g(x)}^{y=f(x)} dx$$

$$= \int_a^b [\underbrace{f(x)}_{top} - \underbrace{g(x)}_{bottom}] \, dx$$

Calc I! (6.1)

(F) Splitting R

p.888



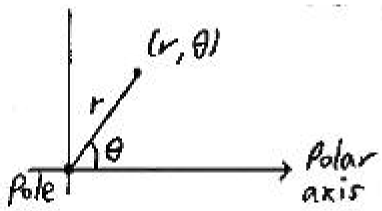
← top changes!

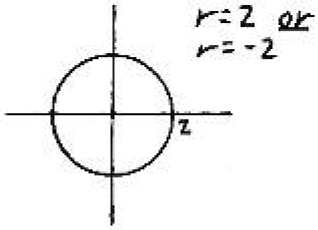
"$R = R_1 \cup R_2$"
↑ union

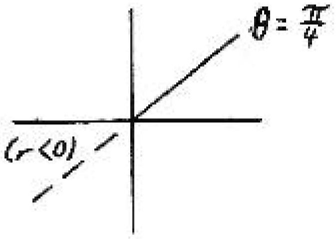
$$\iint_R f(x,y) \, dA = \iint_{R_1} f(x,y) \, dA + \iint_{R_2} f(x,y) \, dA$$

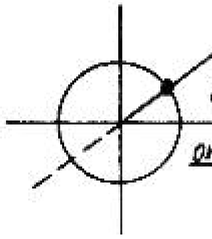
17.3: ∬s IN POLAR COORDS. (PCs)

(A) Review PCs







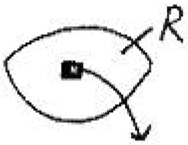


$P\left(2, \frac{\pi}{4} + 2\pi n\right)$ for any integer n

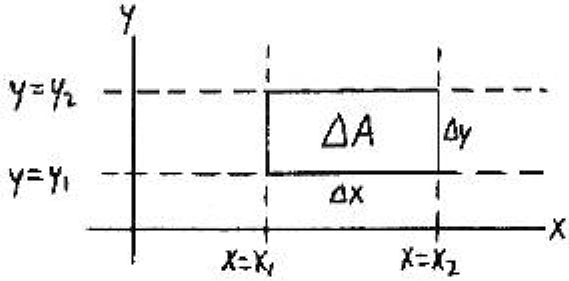
or $P\left(-2, \frac{5\pi}{4} + 2\pi n\right)$ ' ,

(B) Area of a Polar Rectangle

(B1) 17.1/17.2: Cartesian Rectangle



(break up into rectangles for a Riemann sum)



In Riemann Sum,

$$\Delta A = \Delta x \, \Delta y$$

In $\iint$,

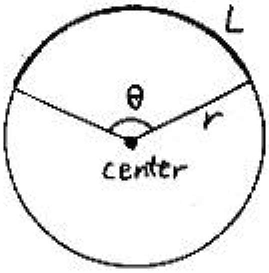
$$dA = dx\, dy$$

or

$$dA = dy\, dx$$

(B2) Arc Length "L" along a Circle

Measure angles in <u>radians.</u>

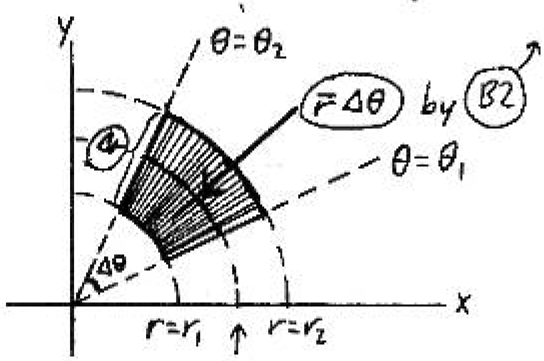


L = (fraction of circle)(circumference)

$$L = \left(\frac{\theta}{\cancel{2\pi}}\right)(\cancel{2\pi} r)$$

$$\boxed{L = r\theta}$$

(B3) Area of a Polar Rectangle



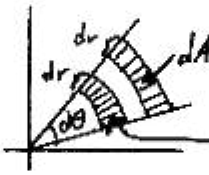
Think: fan/ windshield wiper (How to Ace)

Turns out (see (B4))

$$\text{Shaded } \Delta A = (\Delta r)(\bar{r}\,\Delta\theta)$$
$$= \bar{r}\,\Delta r\,\Delta\theta$$

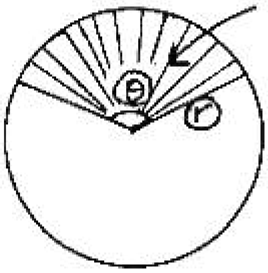
In $\iint$, $\boxed{dA = r\,dr\,d\theta}$

Don't forget!! Idea: If $d\theta$, dr fixed,



Larson 6ed
p. 937

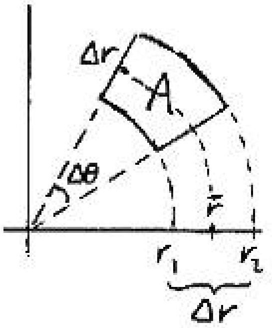
(B4) Optional: Why does $\Delta A = (\Delta r)(\bar{r}\Delta\theta) = \bar{r}\Delta r\Delta\theta$?



Sector Area = (fraction of circle)(area of circle)

$$= \left(\frac{\theta}{2\pi}\right)(\pi r^2)$$

$$= \frac{1}{2} r^2 \theta$$



$$A = \frac{1}{2} r_2^2 \Delta\theta - \frac{1}{2} r_1^2 \Delta\theta$$

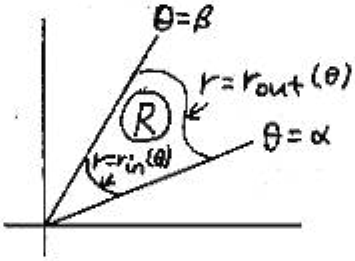
$$= \frac{1}{2} \underbrace{(r_2^2 - r_1^2)}_{\text{Factor}} \Delta\theta$$

$$= \underbrace{\frac{1}{2}(r_2 + r_1)}_{\bar{r}} \underbrace{(r_2 - r_1)}_{\Delta r} \Delta\theta$$

$$= \bar{r}\Delta r \Delta\theta$$

Ⓒ Area

Larson 6ed p.936



Assume

r_{out}, r_{in} are cont., ≥ 0 on $[\alpha, \beta]$

where $\underbrace{\underbrace{0 \leq \beta-\alpha}_{\text{so } \alpha \leq \beta} \leq 2\pi}_{\text{so no "overlapping"}}$

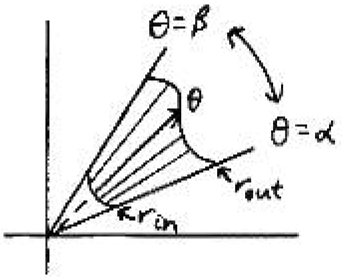
$$\text{Area of } R = \iint_R dA$$

$$= \int_{\theta=\alpha}^{\theta=\beta} \int_{r=r_{in}(\theta)}^{r=r_{out}(\theta)} r\,dr\,d\theta$$

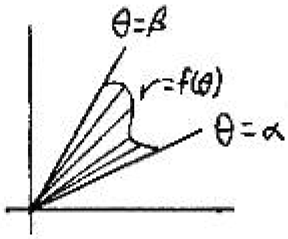
<u>Idea</u>: Aim / Fix θ (outer variable)

Shoot from $r=r_{in}(\theta)$ to $r=r_{out}(\theta)$

Slide / Rotate from $\theta=\alpha$ to $\theta=\beta$



Special Case



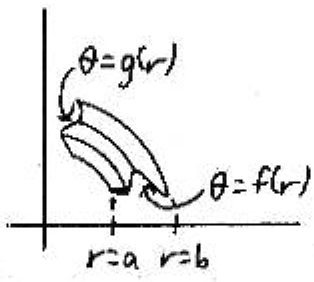
$$\text{Area} = \int_{\theta=\alpha}^{\theta=\beta} \int_{r=0}^{r=f(\theta)} r\,dr\,d\theta$$

$$= \int_{\theta=\alpha}^{\theta=\beta} \left[\tfrac{1}{2} r^2\right]_{r=0}^{r=f(\theta)} d\theta$$

$$= \int_{\theta=\alpha}^{\theta=\beta} \tfrac{1}{2} [f(\theta)]^2\, d\theta$$

Calc II (13.4)

Also



$$\text{Area} = \int_{r=a}^{r=b} \int_{\theta=f(r)}^{\theta=g(r)} r\,d\theta\,dr$$

D Volumes, Other ∫∫s

Ex Find the volume of the solid

bounded above by the graph of $\underbrace{f(x,y)}_{z} = \frac{y^2}{x^2+y^2}$,

bounded below by the xy-plane, and

lying above R,

where R is the region in Quadrant I [of the xy-plane] bounded by the graphs of $x=0$, $y=0$, $y=\sqrt{9-x^2}$, and $y=\sqrt{4-x^2}$.

Sol'n

1 Express $f(x,y)$ in PCs

Recall

$$x = r\cos\theta$$
$$y = r\sin\theta$$

$$x^2+y^2=r^2$$

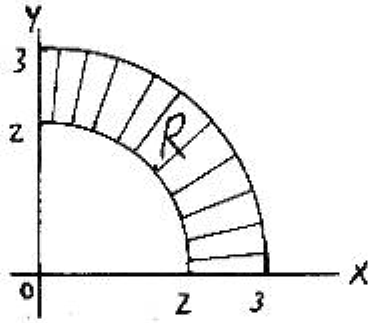
$$\underbrace{\frac{y^2}{x^2+y^2}}_{} = \frac{(r\sin\theta)^2}{r^2} = \frac{\cancel{r^2}\sin^2\theta}{\cancel{r^2}} = \underbrace{\sin^2\theta}_{} \quad (\text{if } r \neq 0)$$

Observe: These ≥ 0 for all $(x,y) \neq (0,0)$ (i.e., $r \neq 0$), so the graph never falls below the xy-plane.

② Graph R (if possible)

$$y = \sqrt{9-x^2} \qquad\qquad y = \sqrt{4-x^2}$$
$$y^2 = 9-x^2, \ y \ge 0 \qquad\qquad y^2 = 4-x^2, \ y \ge 0$$
$$x^2+y^2 = 9, \ y \ge 0 \qquad\qquad x^2+y^2 = 4, \ y \ge 0$$

Improper $\iint$ $r \to 0$



in Quadrant I

part of annulus (ring) ◎

Note These are hard:

$$\int_{x=0}^{x=2}\int_{y=\sqrt{4-x^2}}^{y=\sqrt{9-x^2}} \frac{y^2}{x^2+y^2}\,dy\,dx + \int_{x=2}^{x=3}\int_{y=0}^{y=\sqrt{9-x^2}} \ ,$$

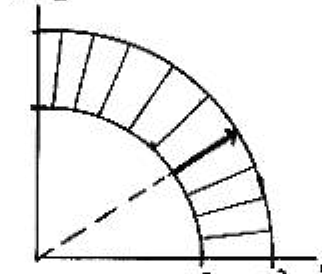
Involves $\tan^{-1}$

$$\int_{y=0}^{y=2}\int_{x=\sqrt{4-y^2}}^{x=\sqrt{9-y^2}} \frac{y^2}{x^2+y^2}\,dx\,dy + \int_{y=2}^{y=3}\int_{x=0}^{x=\sqrt{9-y^2}} \ ,$$

③ Use ② to Set Up the $\iint$ in PCs.

Method 1

$\theta = \frac{\pi}{2}$



r=2 r=3 θ=0

Aim/Fix θ (outer variable).

Shoot From $r=2$ to $r=3$.

Slide/Rotate From $\theta=0$ to $\theta=\frac{\pi}{2}$.

$$\iint_R \sin^2\theta \, dA$$

$$= \int_{\theta=0}^{\theta=\frac{\pi}{2}} \int_{r=2}^{r=3} \underbrace{\sin^2\theta \cdot r}\, dr\, d\theta$$

Indep. of θ; Can separate θ, r into different factors

$\Rightarrow$ We can separate the $\int$s !!

$$= \left[\int_{\theta=0}^{\theta=\frac{\pi}{2}} \sin^2\theta \, d\theta\right] \underbrace{\left[\int_{r=2}^{r=3} r\, dr\right]} \quad \text{☆}$$

④ Evaluate

$$= \left[\frac{r^2}{2}\right]_2^3$$

$$= \frac{(3)^2}{2} - \frac{(2)^2}{2}$$

$$= \frac{5}{2}$$

"sin is bad"

$$= \frac{5}{2}\int_0^{\frac{\pi}{2}} \underbrace{\frac{1-\cos(2\theta)}{2}}\, d\theta$$

from a Power-Reducing Identity (PRI)

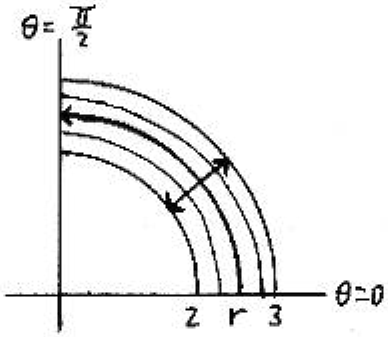
$$= \frac{5}{4}\int_0^{\frac{\pi}{2}} [1 - \underbrace{\cos(2\theta)}_{\text{Use Guess-and-✓ (or u-sub)}}]\, d\theta$$

$$= \frac{5}{4}\left[\theta - \frac{1}{2}\sin(2\theta)\right]_0^{\frac{\pi}{2}}$$

$$= \frac{5}{4}\left(\left[\frac{\pi}{2} - \underbrace{\frac{1}{2}\sin\left(\cancel{2}\cdot\frac{\pi}{\cancel{2}}\right)}_{=0}\right] - [0]\right)$$

$$= \boxed{\frac{5\pi}{8} \text{ cubic units}}$$

Method 2



Aim/Fix r (outer variable).

Shoot from $\theta = 0$ to $\theta = \frac{\pi}{2}$.

Slide from $r = 2$ to $r = 3$.
Push out

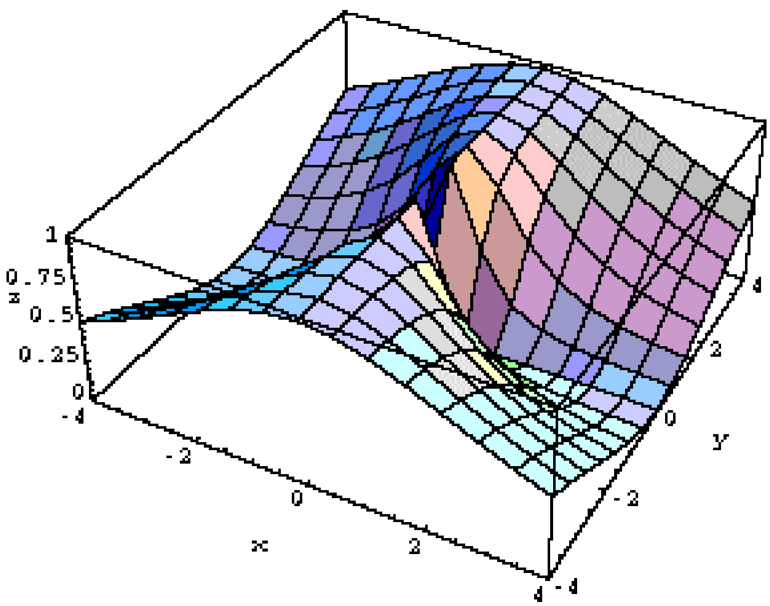
$$\int_{r=2}^{r=3}\int_{\theta=0}^{\theta=\frac{\pi}{2}} \sin^2\theta \cdot r \, d\theta\, dr$$

$= ☆$, also

17.3.11

Here is the graph of $f(x,y)=\dfrac{y^2}{x^2+y^2}$. (Mathematica)

The coordinate axes are rotated a bit differently from what we're used to.



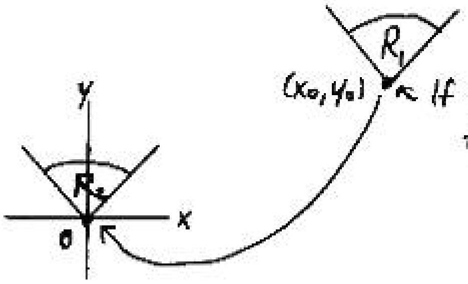
Ⓔ PCs May Help if...

Remember 16.2 on limits

① You see "x^2+y^2" in the integrand

② R is bounded by circular arcs or rays (or other polar curves such as cardioids, limaçons, lemniscates, roses, ...)

Note



If this is not the origin/pole, try translating?

We're finding a volume under a surface. You can translate R if you...

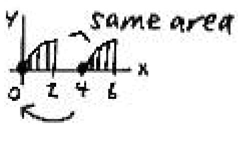
We must also translate the surface(s).

$$\iint_{R_1} f(x,y)\,dA = \iint_{R_2} \underbrace{f(x+x_0,\, y+y_0)}\,dA$$

In the $f(x,y)$ rule, replace:
x with $(x+x_0)$ and
y with $(y+y_0)$.

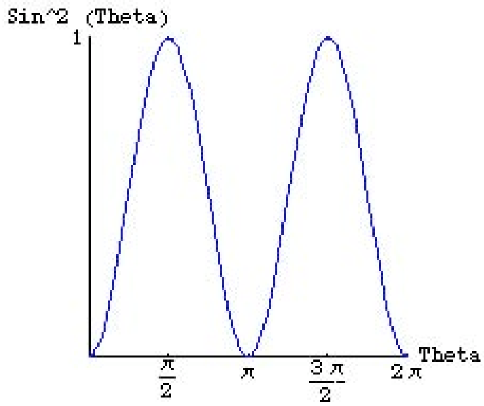
Not "$-$":
O will be our new reference point.

Calc I Idea: $\int_4^6 \underbrace{\sqrt{x-4}}_{f(x)}\,dx = \int_0^2 \underbrace{\sqrt{x}}_{f(x+4)}\,dx$



How do you graph the surface $z = \sin^2 \theta$ over the annulus R from our example in the 17.3 notes? How would you graph the corresponding solid whose volume we were finding?

First off, let's consider the graph of $\sin^2 \theta$ vs. θ in Cartesian coordinates:

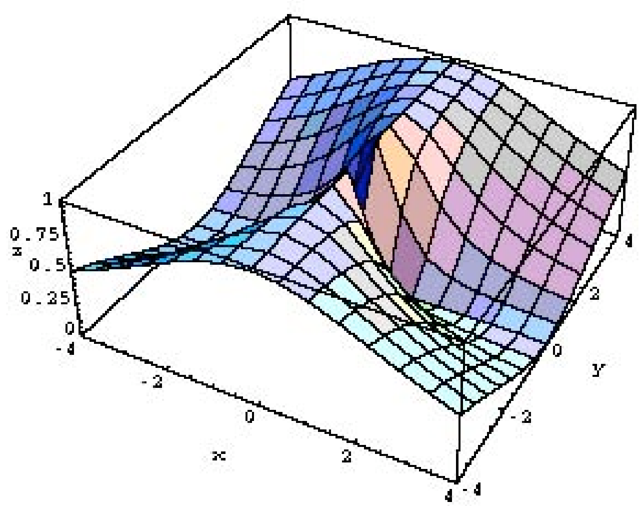


The square of a real number in [0,1] will also be in [0,1].
Because the range of $\sin\theta$ is [0,1], the range of $\sin^2 \theta$ is also [0,1].

Unlike the graph for $\left|\sin\theta\right|$, there are no corners at θ-values of 0, π, 2π, etc. The $\sin^2 \theta$ function is everywhere differentiable! Its derivative is given by $2\sin\theta\cos\theta$, or $\sin(2\theta)$, which is 0 at $\theta = \pi n$ $(n \text{ integer})$.

How do you graph $z = \sin^2 \theta$ in 3-space?

Here's what *Mathematica* gives; the coordinate axes are rotated a bit differently from what we're used to.

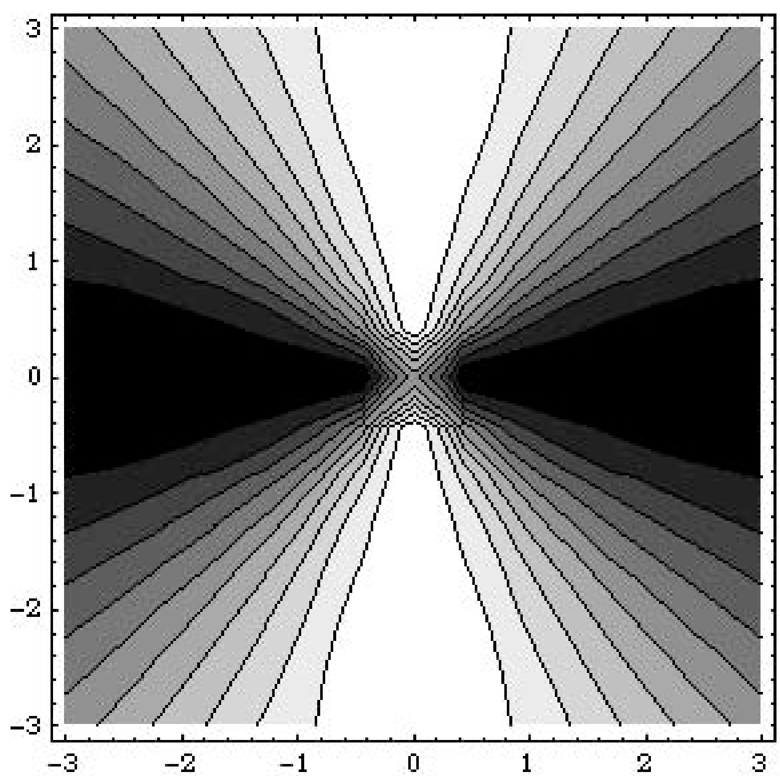


How do we get the piece over R?

You could take scissors to the previous graph, and rotate the scissors as you cut.

It's basically like a twisty slide, or a piece of a roller coaster. Imagine a staircase in a mansion that is curving upward, except that we smooth out the steps. Notice that the function values are constant along the straight lines in the xy-plane through the origin: when θ is fixed, it doesn't matter what r is. The steps are flat, because our function is independent of r and therefore doesn't care about r; it's like sweeping through r-values. The level curves are line segments pointing away from the origin, and they vary from an f or z value of 0 to a value of 1.

Mathematica gives the following Contour Plot; ignore the curviness of some of the lines – these are distortions.



When $\theta = 0$, for example, $\sin^2\theta$ = 0, all the way from $r = 2$ to $r = 3$. That means that the line segment from (2,0,0) to (3,0,0) in Cartesian coordinates is going to be on our slide / staircase. In fact, it will be the bottom edge of our staircase.

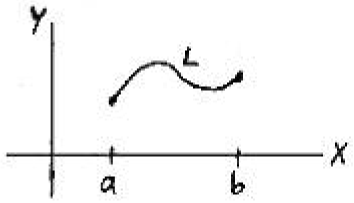
When $\theta = \frac{\pi}{2}$, $\sin^2\theta$ = 1, all the way from $r = 2$ to $r = 3$. That means that the line segment from (0,2,1) to (0,3,1) in Cartesian coordinates is going to be on our slide. In fact, it will be the top edge of our staircase.

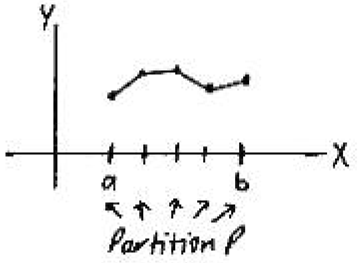
As θ increases from 0 to $\frac{\pi}{2}$, $\sin^2\theta$ increases from 0 to 1 in a curvy way, like the way we discussed in class. $\sin^2\theta$ gives us the *z*-coordinate of our step on the staircase. We don't get any hills or valleys along the staircase, though, because $\sin^2\theta$ is always increasing between $\theta = 0$ and $\theta = \frac{\pi}{2}$. If we go beyond $\frac{\pi}{2}$, though, then the staircase begins to go down.

Look at the first *Mathematica* graph. The top edge of the staircase lies on that top "crease", though it's not really a sharp crease (no corner in our first graph!). The solid whose volume we're finding is basically the wall beneath the staircase.

17.4: SURFACE AREA

(A) Idea: Arc Length, "L" (6.5)



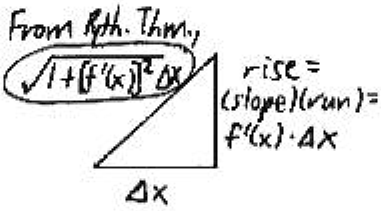


Riemann approx.:

Sample points.
Connect the dots.

We get a piecewise linear frankenstein's monster of pieces of secant lines.

Differentials Idea



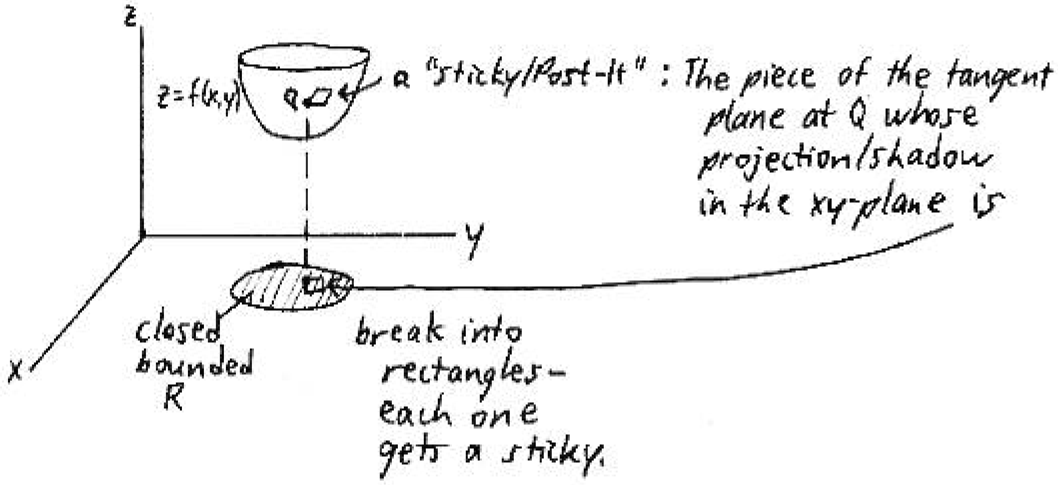
Take the sum of these lengths.

Make partition finer so $\underbrace{\text{widest width}}_{\|P\|\text{, the norm of partition "}P\text{"}} \longrightarrow 0.$

$$\boxed{L = \int_a^b \sqrt{1+[f'(x)]^2}\,dx}$$

Note: This does not depend directly on $f(x)$:

Ⓑ Idea: Surface Area, "S"

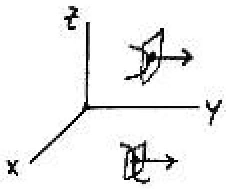


Take the sum of the areas of the stickies.

Make partition finer so $\underbrace{\text{longest diagonal}}_{\|P\|} \to 0$.

Assume f_x, f_y cont. on R
no normal vector to a sticky is $\parallel$ to xy-plane

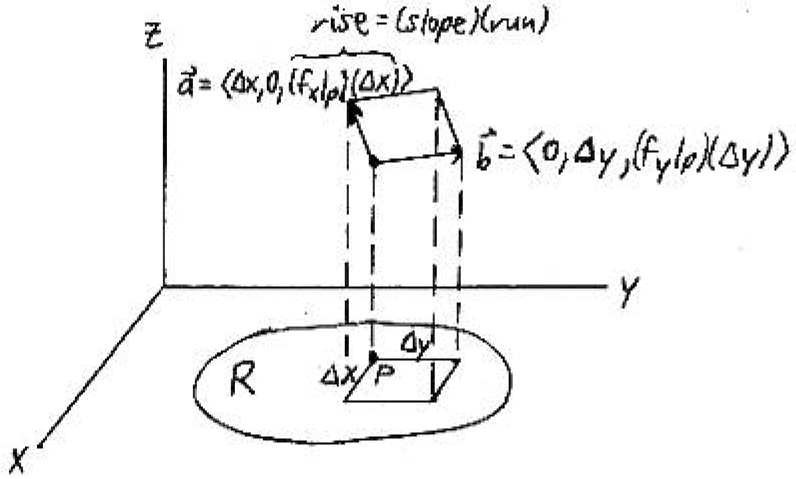
<u>NO</u>



NO $z=f(x,y)$,
could shift perspective

Ⓒ What's the Area of a Sticky?

Again, differentials idea



$$\text{Area} = \|\vec{a} \times \vec{b}\|$$

$$= \left\| \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \Delta x & 0 & (f_x|_P)(\Delta x) \\ 0 & \Delta y & (f_y|_P)(\Delta y) \end{vmatrix} \right\|$$

$$= \| \langle -(f_x|_P)\Delta x \Delta y, -(f_y|_P)\Delta x \Delta y, \Delta x \Delta y \rangle \|$$

$$= \| \langle -(f_x|_P), -(f_y|_P), 1 \rangle \| \Delta x \Delta y$$

I skip in class.

$$= \sqrt{1 + (f_x|_P)^2 + (f_y|_P)^2}\ \Delta x \Delta y$$

(D) Formula for S

$$\boxed{S = \iint_R \underbrace{\sqrt{1 + [f_x(x,y)]^2 + [f_y(x,y)]^2}\ dA}_{= dS}}$$

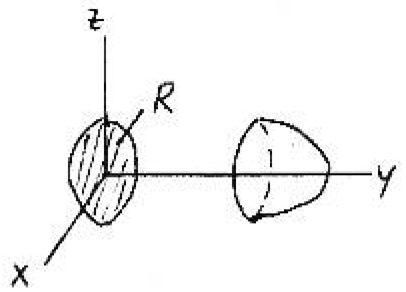
Notes ① Looks like extension of arc length to higher dim.

(I don't know if we deal w/ "surface area" in even higher dims.)

② This formula does not depend directly on f, just like for L. same S

③ Can be modified for, say, $y = f(x,z)$.

May need improper $\iint$. See Challenge Problem



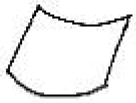
④ If you have a surface of revolution, the Method from 6.5 may be easier. Translations / Rotations may help.

Part of sphere more promising using 6.5 than part of paraboloid, for example.

(I think ellipsoids can be tricky either way.)

Using 17.4, you often use PCs.

Ⓔ Ex

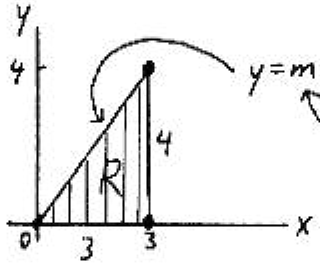


tilted parabolic cylinder

Find the surface area of the graph of $z = x^2 + 4y + 1$ over R, where R is bounded by a triangle with vertices $(0,0)$, $(3,0)$, and $(3,4)$.

Sol'n

R:



$y = \underbrace{m}_{\frac{4}{3}}x + \underbrace{b}_{0}$ $\qquad y = \frac{4}{3}x$

$$f(x,y) = x^2 + 4y + 1$$

$$f_x(x,y) = 2x$$
$$f_y(x,y) = 4$$

$$S = \iint_R \sqrt{1 + [f_x(x,y)]^2 + [f_y(x,y)]^2}\, dA$$
$$= \int_{x=0}^{x=3} \int_{y=0}^{y=\frac{4}{3}x} \sqrt{1 + [2x]^2 + [4]^2}\, dy\, dx$$
$$= \int_{x=0}^{x=3} \int_{y=0}^{y=\frac{4}{3}x} \sqrt{4x^2 + 17}\, dy\, dx$$

If we had $\int \sqrt{4x^2+17}\, dx$, use
Table of $\int$s (pp. A21 to A26)
or Trig Sub: $2x = \sqrt{17}\tan\theta$

Tough $\int$? Try: Table of $\int$s (9.7) or Trig Sub (9.3)
Other Ch. 9 Methods: (Parts (9.1), Neat u-sub (9.1), ...)
PCs (is R well-suited?)

$$\int_{x=0}^{x=3} \left[\sqrt{4x^2+17} \cdot y \right]_{y=0}^{y=\frac{4}{3}x} dx$$

$$= \int_{x=0}^{x=3} \left(\sqrt{4x^2+17} \cdot \frac{4}{3}x - \cancel{[0]} \right) dx$$

$$u = 4x^2+17 \qquad \begin{cases} x=0 \Rightarrow u=17 \\ x=3 \Rightarrow u=53 \end{cases}$$

$$du = 8x\,dx$$

$$\Rightarrow \frac{1}{8}du = x\,dx$$

$$= \int_{17}^{53} \sqrt{u} \cdot \frac{\cancel{4}^{1}}{3} \left(\frac{1}{\cancel{8}_{2}} du \right)$$

$$= \frac{1}{6} \int_{17}^{53} u^{1/2}\, du$$

$$= \frac{1}{6} \left[\frac{u^{3/2}}{3/2} \right]_{17}^{53}$$

$$= \frac{1}{\cancel{6}_{3}} \cdot \frac{\cancel{2}^{1}}{3} \left[u^{3/2} \right]_{17}^{53}$$

$$= \frac{1}{9} \left(53^{3/2} - 17^{3/2} \right)$$

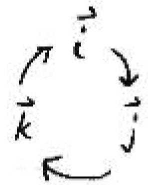
$$\approx \boxed{35.08 \text{ sq. units}}$$

17.5: TRIPLE INTEGRALS ($\iiint$s)

Ⓐ Rotating the Coordinate Axes

Like TNB frame from 15.3

(14.4)



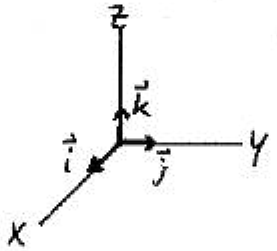
Turn the wheel!

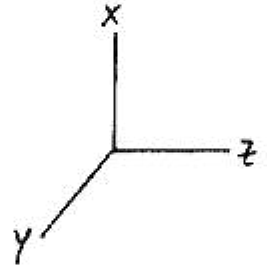
$\vec{i} \times \vec{j} = \vec{k}$ ← thumb from Right-Hand Rule

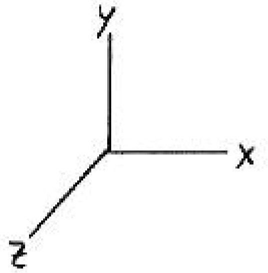
$\vec{j} \times \vec{k} = \vec{i}$

$\vec{k} \times \vec{i} = \vec{j}$

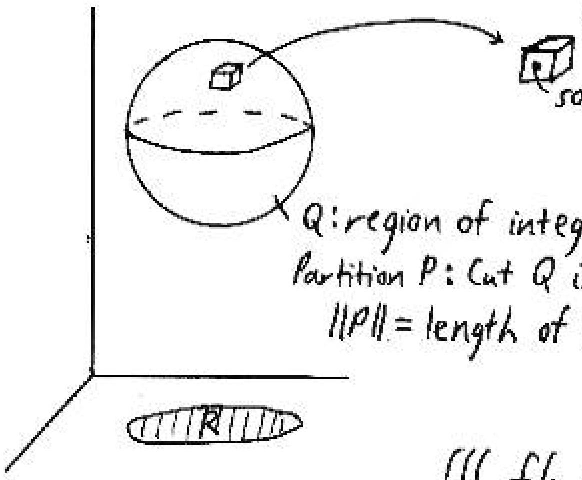
+++ (Octant I) opens towards you







Ⓑ Idea



Volume $= \Delta V_k$

some point (x_k, y_k, z_k) in the box (#k)

Q: region of integration

Partition P: Cut Q into n boxes.

$\|P\|$ = length of longest diagonal

$$\iiint_Q f(x,y,z)\,dV = \lim_{\|P\| \to 0} \sum_{k=1}^{n} \underbrace{f(x_k, y_k, z_k)}_{\text{Assume constant throughout box \#k.}} \Delta V_k$$

Special Case: $\iiint_Q 1\,dV$ = Volume of Q

Ex

Find $\iiint\limits_Q y\,dV$, where Q is the solid [region] bounded by the graphs of:

You observe

① $y=0$

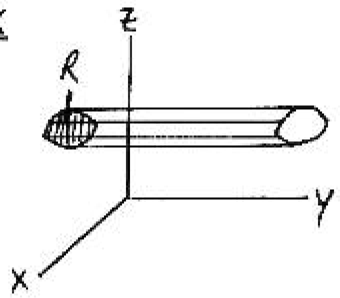
② $y-x^2=1$

③ $z=x^2-3x$

④ $2x+z=2$

$\Big\}$ Cylinders $\perp$ xz-plane (swept $\parallel$ y-axis)
We hope they "trap" some space with projection R on the xz-plane.

Ex



Sol'n

Step 1 What is R?

Step 1a Intersection Points

$$\begin{cases} ③ & z=x^2-3x \\ ④ & 2x+z=2 \Rightarrow z=2-2x \end{cases} \Rightarrow \begin{aligned} & x^2-3x=2-2x \\ & x^2-x-2=0 \\ & (x-2)(x+1)=0 \end{aligned}$$

Optional: Use $z=2-2x$

$$\Rightarrow x=2 \text{ or } x=-1 \quad \begin{pmatrix} \Rightarrow z=-2 \\ \Rightarrow z=4 \end{pmatrix}$$

If do $\overset{z}{+}x$, θ orientation reversed; y points into board

↓counter

↑clockwise

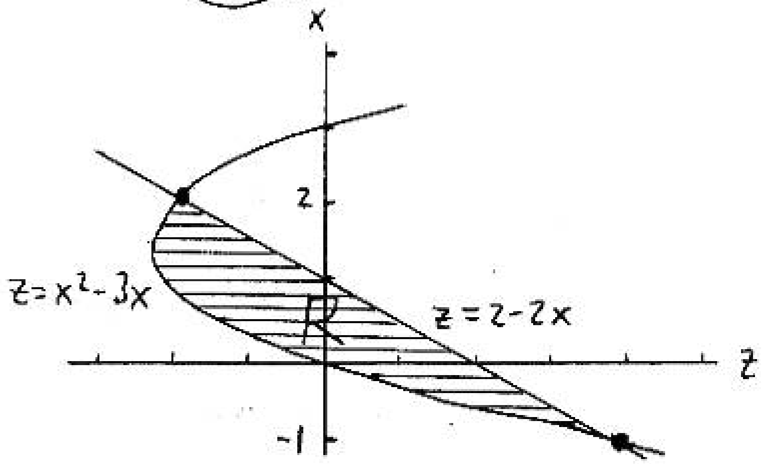
Step 1b In $\overset{x}{+}z$, which graph is on right? left?

(3) $z = x^2 - 3x$ ⇒ parabola opening right ⊂ ← left

(4) $2x + z = 2$ ⇒ line \\ ← right

↑ they intersect, so →

Turns out Graph R?



Easier to scan this way (dx: outer)

Step 2 Based on R, Set Up the Outer ∬

$$\int_{x=-1}^{x=2} \int_{z=x^2-3x}^{z=2-2x} \left[\int_{y=?}^{y=??} y \, dy \right] dz \, dx$$

Visualize?

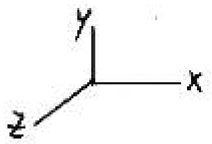
Use symmetry? (Take integrand into account; also inner ∫s.)

Use PCs?

$x^2 + z^2 = r^2$, $z = r\cos\theta$, $x = r\sin\theta$

Step 3 $\underline{\text{Innermost} \int}$

① $y = 0$ (bottom)



② $y - x^2 = 1 \Rightarrow y = x^2 + 1 \geq 0$ for all x in $[-1, 2]$
(top)

$$\int_{x=-1}^{x=2} \int_{z=x^2-3x}^{z=2-2x} \int_{y=0}^{y=x^2+1} y \, dy \, dz \, dx$$

← (the Set-Up)

Step 4 Evaluate

$$= \int_{x=-1}^{x=2} \int_{z=x^2-3x}^{z=2-2x} \left[\frac{y^2}{2} \right]_{y=0}^{y=x^2+1} dz \, dx$$

$$= \int_{x=-1}^{x=2} \int_{z=x^2-3x}^{z=2-2x} \frac{1}{2}\left[(x^2+1)^2 - \cancel{(0)}\right] dz \, dx$$

$$= \frac{1}{2} \int_{x=-1}^{x=2} \left[(x^2+1)^2 z\right]_{z=x^2-3x}^{z=2-2x} dx$$

$$= \frac{1}{2} \int_{-1}^{2} \left(\left[(x^2+1)^2(2-2x)\right] - \left[(x^2+1)^2(x^2-3x)\right] \right) dx$$

Calc I !!

⋮

$$= \boxed{\frac{225}{28}}$$

① Mass

<u>Old Ex</u> $\iiint_Q \underbrace{y^2}\, dV$ = Mass of Q (i.e., the solid taking up Q)

Let's say this is $\rho(x, y, z)$, or $\delta(x, y, z)$,
a <u>mass density function</u>.
Nonconstant on $Q \Rightarrow$ Solid is <u>nonhomogeneous</u>.

(E) Average Value of f in Q

temperature!

$$= \frac{\iiint_Q f(x,y,z)\, dV}{\iiint_Q dV} \quad \leftarrow \text{Volume of } Q$$

This follows the classic "average" template of

$$\frac{\text{Sum}}{\text{Input Size}} \quad \leftarrow \text{"Integration is continuous summation."}$$

17.6: MOMENTS and CENTERS OF MASS

(A) Idea: Weighted Averages (Discrete Case)

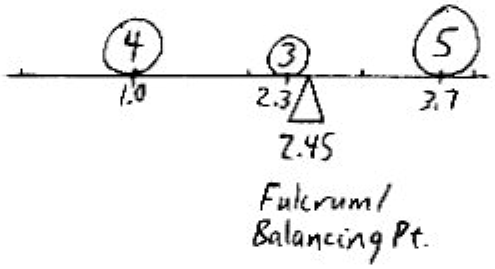
Ex (GPA)

	Grade (x)	# Units (Weights w)
Math	A- (3.7)	5
Chem	D (1.0)	4
Bio	C+ (2.3)	3
		12

$$GPA = \frac{\sum x \cdot w}{\sum w}$$

$$= \frac{(3.7)(5) + (1.0)(4) + (2.3)(3)}{12}$$

$$= 2.45$$



(B) 2D Lamina "L" with Shape R: Center of Mass

(Really, there should be some [constant] thickness.)

Mass of L = "m"

$$= \iint_R \underbrace{\delta(x,y)}_{\text{or } \rho(x,y)\text{: area mass density at } (x,y)} \, dA$$

or $\rho(x,y)$: area mass density at (x,y)
(mass per unit area)

Center of Mass = $(\bar{x}, \bar{y})$, where

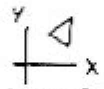
$$\bar{x} = \frac{\overbrace{\iint_R x\,\delta(x,y)\,dA}^{M_y}}{\underbrace{\iint_R \delta(x,y)\,dA}_{m}}$$

M_y, the first moment about the y-axis. Why? (*)

This is [literally] a weighted average of the x-coords. throughout R.

Think: x = grade
$\delta(x,y)$ = # units

Exs



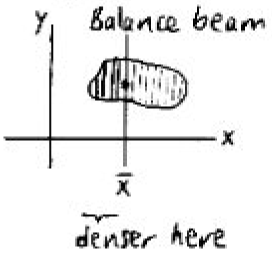
shape favors higher $\bar{x}$, but what about density?

density favors lower $\bar{x}$

(*) M_y measures the tendency to rotate about the y-axis.

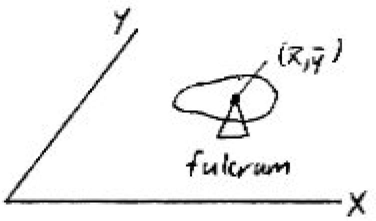
Larson 6.6

1-D Ex: If you want to rotate a see-saw, would you sit close to the fulcrum or far away?



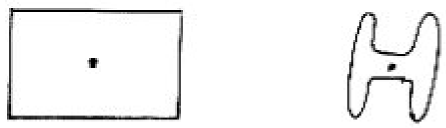
(Imagine mass of L being concentrated at the center of mass.)

$$\bar{y} = \frac{\overbrace{\iint_R y\,\delta(x,y)\,dA}^{M_x,\ \text{the first moment about the x-axis.}}}{\underbrace{\iint_R \delta(x,y)\,dA}_{m}}$$



If $\delta(x,y)$ = a constant, then L is <u>homogeneous</u>, and $(\bar{x},\bar{y})$ is the <u>centroid</u>, which only depends on shape. (See Section 6.7.)

<u>Ex</u>

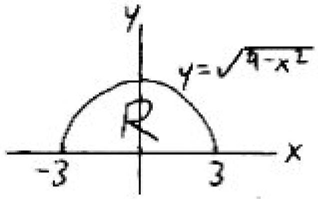


Use symmetry!

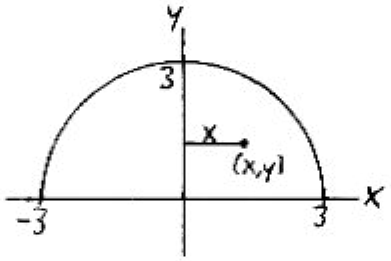
<u>Ex</u> L has shape R, which is bounded by the x-axis and the graph of $y = \sqrt{9-x^2}$. The density at $P(x,y)$ is directly proportional to the square of the distance from P to the y-axis. Find the center of mass for L.

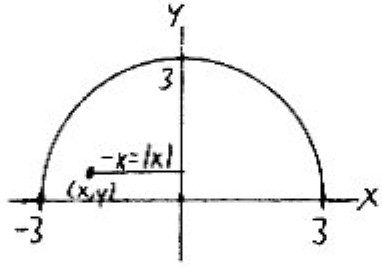
<u>Sol'n</u>

<u>Sketch R</u>



Find Density $\delta(x,y)$





Distance from P to the y-axis $= |x|$
Square of this $= x^2$

$\delta(x,y) = kx^2$, where $k > 0$
constant of proportionality

(don't have to find)

Guess!

Find $\bar{x}$

How to Ace 208

$\delta(x,y) = kx^2$ ← "even in x": $\delta(-x,y) = \delta(x,y)$
$k(-x)^2 = kx^2$
True even if x were missing.

Symmetry of δ about y-axis $(x=0)$.

and

R is symmetric about the y-axis $(x=0)$.

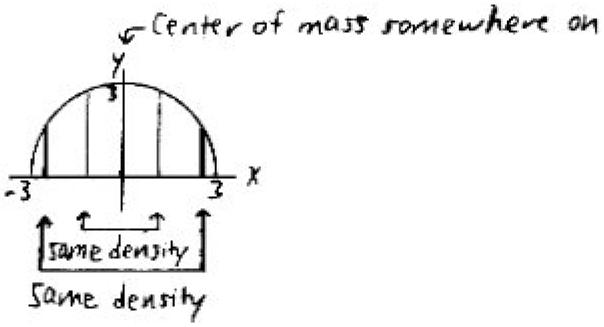
Observe:

$$\bar{x} = \frac{\iint_R \overbrace{x(kx^2)}^{\text{odd in } x}\, dA}{m} = 0$$

Sym.

$\Rightarrow \bar{x} = 0$

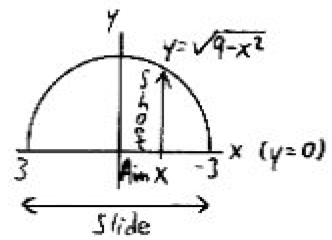
Idea



Find $\bar{y}$

$$\bar{y} = \frac{\iint_R y\,\delta(x,y)\,dA}{\iint_R \delta(x,y)\,dA}$$

R:



$$= \frac{\int_{x=-3}^{x=3}\int_{y=0}^{y=\sqrt{9-x^2}} y\cdot kx^2\,dy\,dx}{\int_{x=-3}^{x=3}\int_{y=0}^{y=\sqrt{9-x^2}} kx^2\,dy\,dx}$$

Set-Up

Don't have to find k!!

$$= \frac{\cancel{k}\int_{x=-3}^{x=3}\left[x^2\cdot\frac{y^2}{2}\right]_{y=0}^{y=\sqrt{9-x^2}}dx}{\cancel{k}\int_{x=-3}^{x=3}\left[x^2 y\right]_{y=0}^{y=\sqrt{9-x^2}}dx}$$

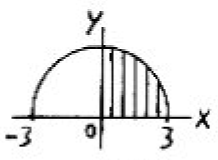
$$= \frac{\int_{x=-3}^{x=3}\frac{1}{2}x^2(\sqrt{9-x^2})^2\,dx}{\int_{x=-3}^{x=3}x^2\sqrt{9-x^2}\,dx}$$

Both integrands are even in x, so $\int_{-3}^{3}\cdots = 2\int_0^3\cdots$

We like to plug in after we find an antiderivative.

$$= \frac{2\int_0^3 \overbrace{\tfrac{1}{2}x^2(9-x^2)}^{\text{Multiply out}}\,dx}{\underbrace{2\int_0^3 x^2\sqrt{9-x^2}\,dx}}$$

Use Trig Sub
or Table of $\int$s (p. A22, #31) ← Do



has same $\bar{y}$ as

$$\vdots$$

$$= \frac{\frac{81}{5}}{\frac{81\pi}{16}}$$

$$= \frac{16}{5\pi}$$

Center of mass $= (\bar{x}, \bar{y}) = \boxed{\left(0, \frac{16}{5\pi}\right)}$

Note: $\frac{16}{5\pi} \approx 1.02$. Why so low?

© 2D "L": Moments of Inertia, or Second Moments: I_x and I_y and I_0

I_x: wrt x ($\underbrace{\text{wrt}}_{\text{with respect to}}$)

I_y: wrt y

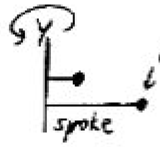
I_0: wrt origin

$$I_y = \iint_R x^2 \,\delta(x,y)\,dA$$

This is a "weighted sum" of squared distances of points from the y-axis.

Larson: Mass is a measure of resistance to straight-line motion (Calc I)

Higher $I_y \Rightarrow$ Harder to rotate/revolve by external force; it measures resistance to angular momentum.



Harder to rotate by force

Cartoon Guide to physics: Moment of inertia is resistance to angular momentum.

$$I_x = \iint_R y^2 \,\delta(x,y)\,dA$$

$$I_0 = I_x + I_y$$
$$= \iint_R \underbrace{(x^2 + y^2)}_{= r^2 \,=\, \text{sq. dist. from } O} \delta(x,y)\,dA$$

Along circle centered at O?

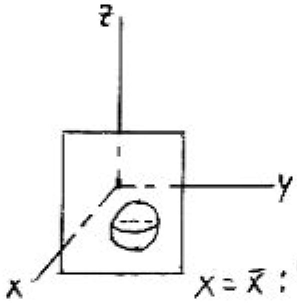
This is the <u>polar moment of inertia</u>.

(D) 3D Solid with Shape "Q": Center of Mass

$(\bar{x}, \bar{y}, \bar{z})$ where

$$\bar{x} = \frac{\overbrace{\iiint_Q x \overbrace{\delta(x,y,z)}^{\text{mass density}} dV}^{M_{yz}\text{, the first moment about the yz-plane } (*)}}{\underbrace{\iiint_Q \delta(x,y,z)\, dV}_{m\text{, the mass of the solid}}}$$

(*) M_{yz} measures the tendency to rotate about the yz-plane.
(Is this visualizable?)

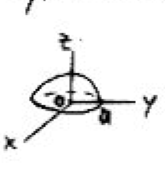


$x = \bar{x}$: "Plane stand" for solid

$\bar{y}, \bar{z}$ analogous

<u>Ex</u> Q: upper hemisphere of radius a w/base centered at O.
$\delta(x,y,z) = \underbrace{y^4 + 3y^2 + z + 1}$

even in x <u>and</u> Q is sym. about yz-plane
$\Rightarrow \bar{x} = 0$

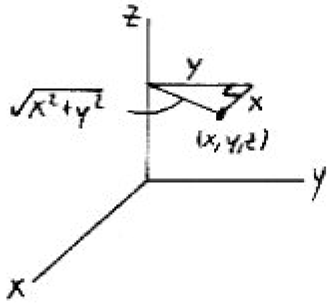


Similarly, $\bar{y} = 0$.

Ⓔ 3D: Moments of Inertia: I_x, I_y, I_z

$$I_z = \iiint_Q (x^2+y^2)\,\delta(x,y,z)\,dV \quad \leftarrow \text{Moment of inertia about the } z\text{-axis}$$

This is a "weighted sum" of squared distances of points from the z-axis.



Higher I_z ⇒ Harder to rotate/revolve by external force

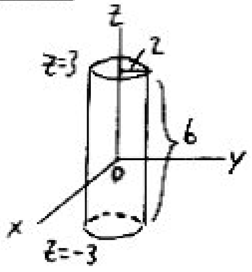
(Think: big wheel$_z$ as opposed to ϕ_z)

I_x, I_y analogous

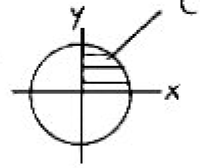
<u>Ex</u> Set up a triple integral for I_y for a cylinder of base radius 2 and height 6 centered at the origin. $\delta(x,y,z) = x^2z^2$.

<u>Sol'n</u>

Q:



R:



Can use symmetry of R about x,y-axes, because $\delta(x,y,z) = x^2z^2$ even in y, x.

$$I_y = 4\int_0^2 \int_0^{\sqrt{4-x^2}} \underbrace{\int_{-3}^{3}}_{\text{Can use sym., } \delta \text{ even in } z:\ 2\int_0^3} (x^2+z^2)\overbrace{x^2z^2}^{\delta(x,y,z)}\,\overbrace{dz\,dy\,dx}^{dV}$$

$I_x = \frac{2184\pi}{5}$

$I_z = 142\pi$

Turns out: $\frac{2664\pi}{5}$

17.7: CYLINDRICAL COORDS. (Cyl.Cs)

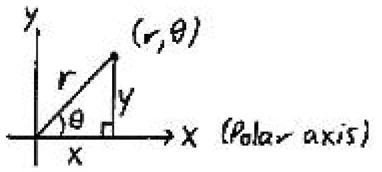
Ⓐ Intro

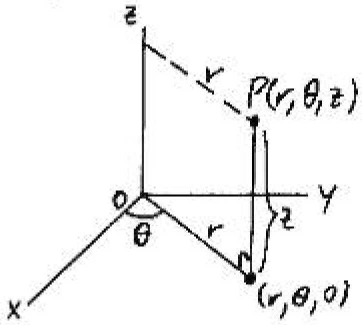
How do we extend PCs to 3D?

Throw in z (or x or y, as the case may be)

if you use PCs to coordinatize the yz-plane

PCs (2D)	Cyl.Cs (3D)





$x = r\cos\theta$

$y = r\sin\theta$

$x^2 + y^2 = r^2$

$\tan\theta = \frac{y}{x}$

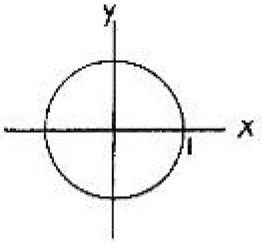
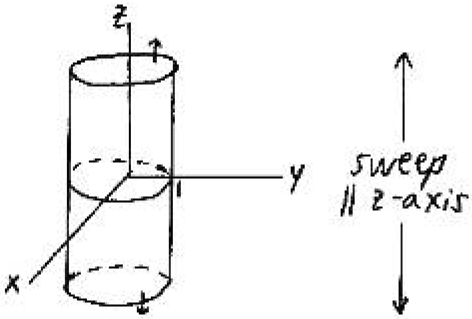
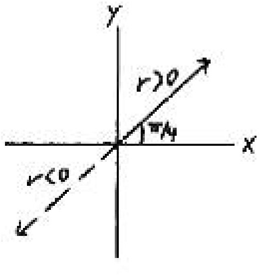
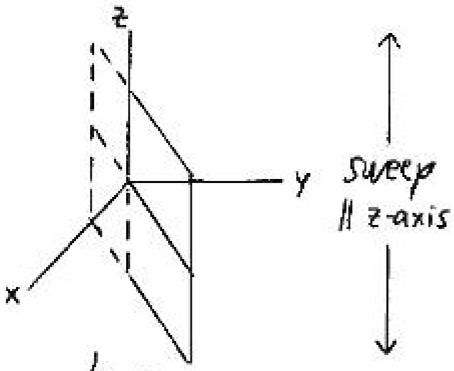
(if $x \neq 0$; Watch Quadrant!)

r = distance of P from z-axis

(B) The Basic Principle of Graphing

> The graph of an equation consists of all points whose coords. satisfy the equation.

Graph:

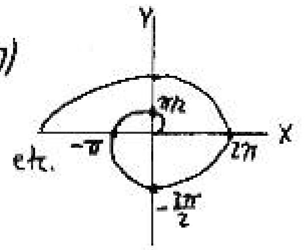
	PCs (2D)	Cyl. Cs (3D)
$r=1$	 circle	 right circular cylinder {points 1 unit away from the z-axis}
$\theta=\frac{\pi}{4}$	 line	 plane

If θ unrestricted over reals $\Rightarrow$ sym. about y-axis

p.661 - Swok

PCs (2D)

$r = \theta$
$(\theta \geq 0)$



a spiral of Archimedes
$r = a\theta,\ a \neq 0$

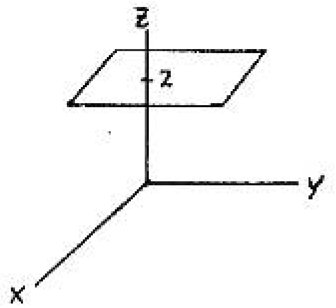
Cyl. Cs (3D)

Sweep $\parallel$ z-axis

Think: Hostess treat
Cinnabon

Graph in (3D):

$z = 2$ (same as for Cartesian Cs)



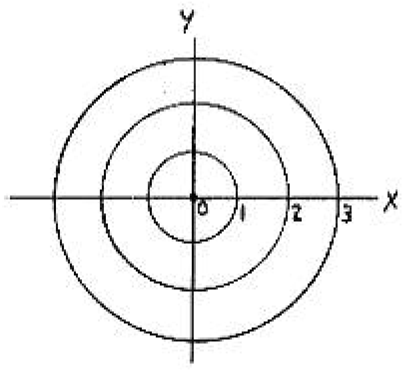
plane

$z = r$ $(r \geq 0)$

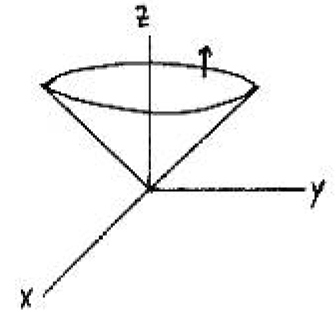
A level curve analysis may help:

$$\underbrace{f(r, \theta)}_{= z} = r$$

Replace with k



Graph:



$z = r$
$\Rightarrow$ Cartesian Cs

$z^2 = r^2 \quad (z \geq 0)$

$z^2 = x^2 + y^2 \quad (z \geq 0)$

Upper nappe of a cone

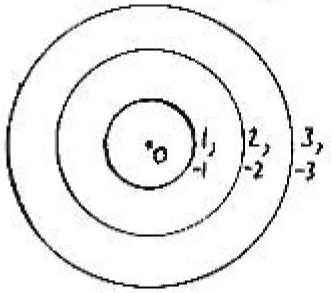
Graph $z = r$ (r unrestricted over reals)

A level curve analysis

"multiple identities"

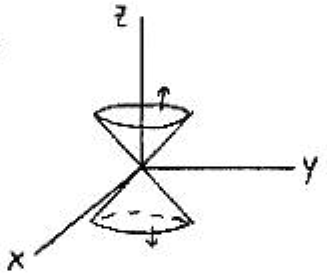
$\underbrace{f(r, \theta)}_{= z} = r$ ← The Vertical Line Test does <u>not</u> apply. (x, y) has multiple PC representations.

LCs <u>can</u> intersect in Cyl. Cs settings.



Lift <u>and</u> drop

Graph:

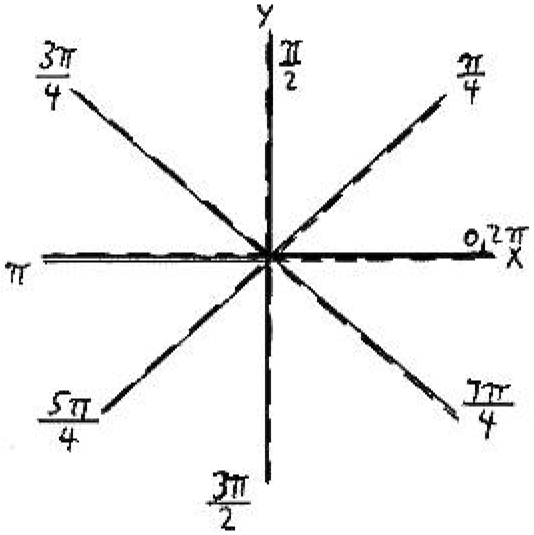


Complete double-napped cone

Graph $z=\theta \quad (0 \leq \theta \leq 2\pi)$

A level curve analysis

$$\underbrace{f(r,\theta)}_{=z}=\theta$$



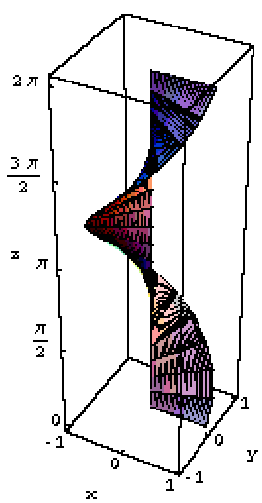
Graph $z = \theta \quad \left(0 \le \theta \le 2\pi\right)$.

Think of a rising, rotating line with a rising z-intercept that never intersects the xy-plane (except for $z = 0$). These lines are "parallel" to the xy-plane.

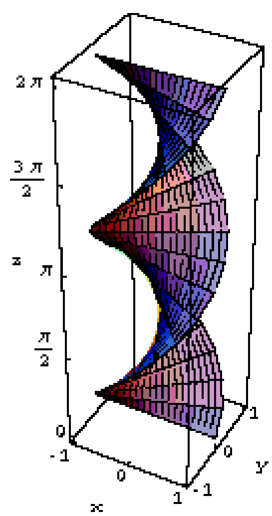
The pole may be coordinatized as $\left(0, \theta\right)$ in polar coordinates, where θ is any real number. Here, we have the restriction $0 \le \theta \le 2\pi$. The multiple representations ("identities") of the pole lead to infinitely many image points along the z-axis of the form $\left(0, \theta = z, z = \theta\right)$, or $\left(0, z, z\right)$, where $0 \le \theta \left(\text{or } z\right) \le 2\pi$.

Mathematica graphs:

$0 \le r \le 1$



$-1 \le r \le 1$



© Ex

Describe the graph of $r\cos\theta = \tan\theta + 4$

<u>Sol'n</u> Let's go to Cartesian Cs.

<u>or</u>

$x = \frac{y}{x} + 4$

$(x \neq 0)$

$x = 0 \Leftrightarrow$ $\tan\theta$ und. (unless $y=0$, also)

$$r\cos\theta = \tan\theta + 4$$

$$r\cos\theta = \frac{\sin\theta}{\cos\theta} + 4$$

$$r\cos^2\theta = \sin\theta + 4\cos\theta \qquad (\cos\theta \neq 0)$$

<u>Trick</u>: Multiply both sides by r.

$$r^2\cos^2\theta = r\sin\theta + 4r\cos\theta \quad (\text{if } \underbrace{\cos\theta \neq 0}, \underbrace{r \neq 0})$$

Note: $\cos\theta = 0$ $\Leftrightarrow$ $\tan\theta$ is undefined, so no corresp. points

if $r = 0$, we get $0 = \tan\theta + 4$. See ④ on 17.7.8.

$$x^2 = y + 4x$$

$$\underbrace{x^2 - 4x}_{\text{Complete the Square}} = y$$

$$(x^2 - 4x + 4) - 4 = y$$

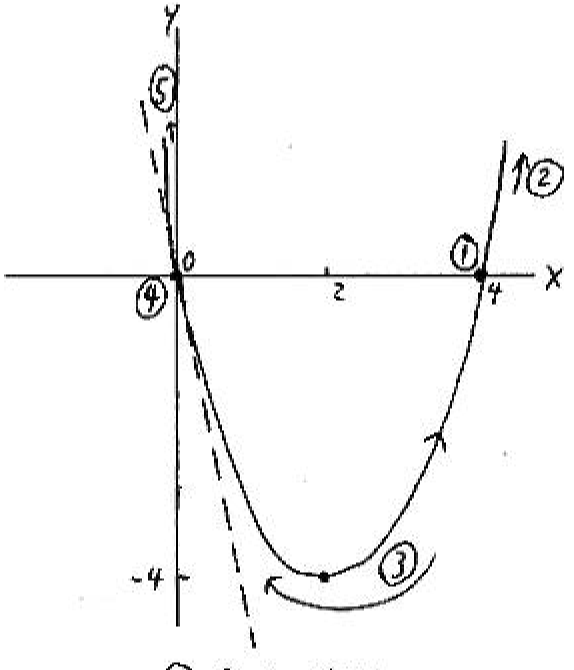
$$y = (x-2)^2 - 4$$

Only z missing $\Rightarrow$ Right cylinder orthogonal to the xy-plane

It's a parabolic cylinder whose xy-trace is an upward-opening ($a = 1 > 0$) parabola with vertex $(2, -4)$.
form: $y = a(x-h)^2 + k$ has vertex (h,k).

(May Skip)

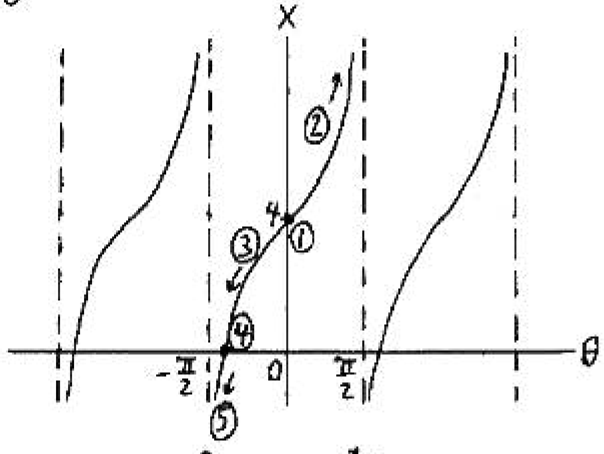
Why does this make sense? Let's focus on the xy-trace.



④ $\theta = \tan^{-1}(-4)$

<u>Note:</u>
$\cos\theta = 0$ at no point, not even the pole.

Graph $\underbrace{x}_{r\cos\theta} = \tan\theta + 4$ using θ and x as Cartesian Cs.



① $\theta = 0 \Rightarrow x = \overbrace{\tan\theta}^{=0} + 4$
$= 4$

② $\theta \nearrow \frac{\pi}{2}^- \Rightarrow x \nearrow \infty$

③ $\theta : 0 \searrow \tan^{-1}(-4)$
$\Rightarrow x : 4 \searrow 0$

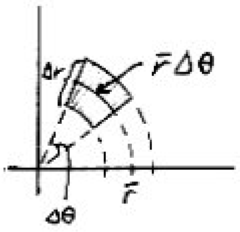
④ $x = 0 \Leftrightarrow \tan\theta = -4$
e.g., $\theta = \tan^{-1}(-4)$

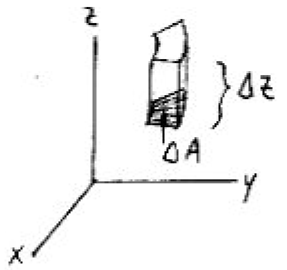
⑤ $\theta \searrow -\frac{\pi}{2}^+ \Rightarrow x \searrow -\infty$

correspond to point at infinity (the parabola is a closed curve passing through this when θ crosses an asymptote; think of wrapping it around a sphere)

(D) Volume, ΔV, of a Box in Cyl. Cs.

(17.3) Area of a Polar Rectangle (Now)





$\Delta A = \bar{r}\Delta r \Delta\theta$

$$\Delta V = \Delta A\, \Delta z$$
$$= \bar{r}\Delta r \Delta\theta \Delta z$$

$$\boxed{dV = r\, dr\, d\theta\, dz}$$

or $r\, dz\, dr\, d\theta$ ← more common?
etc.

(E) Ex (#33)

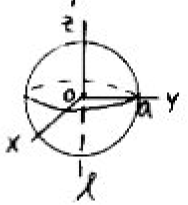
A spherical solid has radius a, and the density at $P(x,y,z)$ is directly proportional to the distance from P to a fixed line ℓ through the center of the solid. Find its mass.

Sol'n

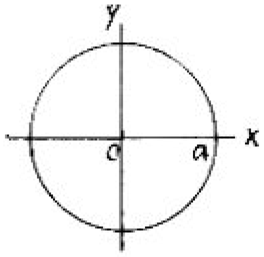
Orient the Sphere "Q"

Let its center be at O.
Let the z-axis be ℓ.

Sketch Q (Optional?)



Sketch R, the projection of Q onto the xy-plane



An easy polar rectangle!

Find density, δ

$$\delta = kr \quad (r \ge 0)$$

More precisely, $\delta(r,\theta,z)$, but remember that a point has multiple representations in Cyl. Cs.

Find mass, m, of the solid

$$m = \iiint_Q \underbrace{kr}_{\substack{\delta \\ \text{func.}}} \underbrace{dV}_{= r\,dr\,d\theta\,dz}$$

Use R to set up the outer $\iint$.

$$= \underbrace{\int_{\theta=0}^{\theta=2\pi}}_{} \int_{r=0}^{r=a} \left[\int_{z=?}^{z=??} kr \, dz \right] r \, dr \, d\theta$$

or $4\int_{\theta=0}^{\theta=\pi/2}$ by sym., but it won't matter

may depend on r, θ

Find z-limits based on surfaces.

Boundary of

$$Q: \quad x^2+y^2+z^2=a^2$$

$$r^2+z^2=a^2$$

$$z^2=a^2-r^2$$

$$z=\pm\sqrt{a^2-r^2}$$

correspond to upper, lower hemispheres

$$m=\int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=a}\left[\int_{z=-\sqrt{a^2-r^2}}^{z=\sqrt{a^2-r^2}} kr\,dz\right] r\,dr\,d\theta$$

$$=k\int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=a} r^2\left[\underbrace{2\int_{z=0}^{z=\sqrt{a^2-r^2}}}_{\text{by sym. of } Q \text{ about } xy\text{-plane } (z=0),\ \delta \text{ even in } z} dz\right] dr\,d\theta$$

$$\underbrace{\hphantom{xxxxxxxx}}$$

$$=2\left[z\right]_{z=0}^{z=\sqrt{a^2-r^2}}$$

$$=2\sqrt{a^2-r^2}$$

$$=2k\int_{\theta=0}^{\theta=2\pi}\underbrace{\int_{r=0}^{r=a} r^2\sqrt{a^2-r^2}\,dr}_{\text{Use Table of }\int\text{s, p. A22, \#31}}\,d\theta$$

$$\overset{(u=r)}{\Rightarrow}\left[\frac{r}{8}(2r^2-a^2)\sqrt{a^2-r^2}+\frac{a^4}{8}\sin^{-1}\left(\frac{r}{a}\right)\right]_{r=0}^{r=a}$$

$$=\left[\underbrace{\frac{a}{8}(2a^2-a^2)\underbrace{\sqrt{a^2-a^2}}_{=0}}_{=0}+\frac{a^4}{8}\underbrace{\sin^{-1}\left(\frac{a}{a}\right)}_{=\sin^{-1}(1)=\frac{\pi}{2}}\right]-[0]$$

Note: $\sin^{-1}(0)=0$

$$=\frac{\pi a^4}{16}$$

17.7.12

$$= \cancel{2}k\left(\frac{\pi a^4}{\cancel{16}_{8}}\right)\underbrace{\int_0^{2\pi} d\theta}$$

$$= [\theta]_0^{2\pi}$$

$$= 2\pi$$

$$= \frac{\pi k a^4}{\cancel{8}_4}\cdot\cancel{2}\pi$$

$$= \boxed{\frac{\pi^2 k a^4}{4}} \text{ [mass units]}$$

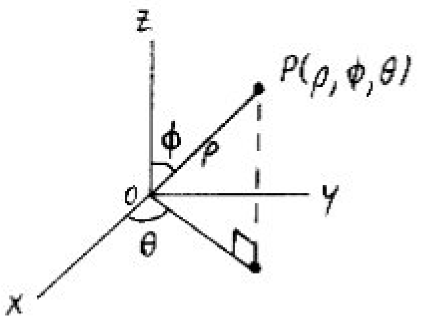
17.8: SPHERICAL COORDS. (SCs)

(A) Intro

How to Ace: same order
Larson: (ρ, θ, ϕ)

Webster's: fī
Everyone else: fee

$\rho \geq 0, \phi \geq 0$ hinder exploitation of symmetry



ρ = distance from 0 to P. ($\rho \geq 0$)
"rho"

ϕ = angle between (nonnegative) positive z-axis, $\overline{OP}$.
"phi" $(0 \leq \phi \leq \pi)$. "Slot machine pullout angle"

θ : same as for Cyl. Cs, but assume $r \geq 0$.

"–" helps w/direction

		# measures of distance	# measures of angles / directions
(x,y,z)	Cartesian Cs	3 } Share z	0
(r,θ,z)	Cyl. Cs	2 }	1 } Share θ
(ρ,ϕ,θ)	SCs	1	2 }

(ρ, ϕ, θ) unique for any pt. off z-axis if θ restricted to $[0, 2\pi)$

ρ, ϕ are unique for P, but θ is not.

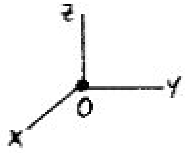
not unique if P is 0; there, ϕ can be anything in $[0, \pi]$

source of multiple representations for P
Along z-axis, θ can be any real #.

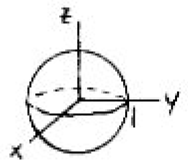
Game: If you fix ρ, ϕ, why does θ locate a point? Etc...
What happens if we "shoot" θ? (SSS issue)

(B) Basic Graphs

$\rho = 0$



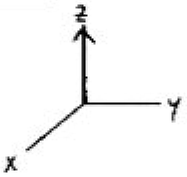
$\rho = 1$



Sphere
(sweep ϕ, θ)

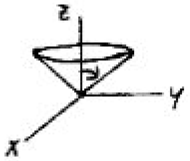
Like a blossoming, then dying flower...

$\phi = 0$



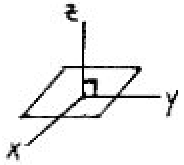
nonnegative
z-axis

$\phi = \frac{\pi}{4}$



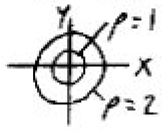
upper nappe
of a cone
"pull out and
spin"

$\phi = \frac{\pi}{2}$

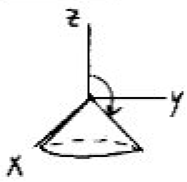


xy-plane
(ρ acts like r)

$\phi = \frac{\pi}{2}$

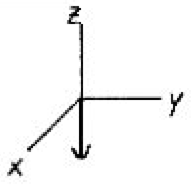


$\phi = \frac{3\pi}{4}$



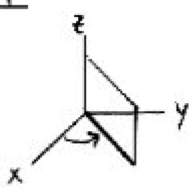
lower nappe
of a cone

$\phi = \pi$



nonpositive
z-axis

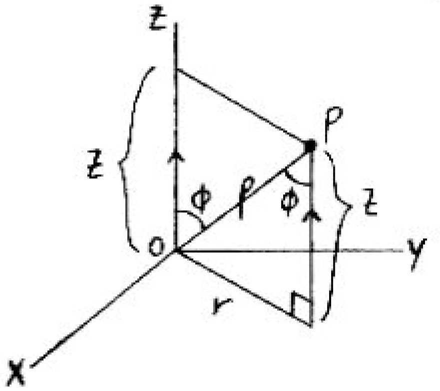
$\theta = \frac{\pi}{4}$



half-plane (full plane for Cyl.Cs)

(C) Conversions

<u>SCs → Cartesian Cs</u> $(\rho, \phi, \theta) \to (x, y, z)$



If $\rho = 0$, ϕ can be anything in $[0, \pi]$.

$\sin\phi = \frac{r}{\rho} \Rightarrow r = \rho \sin\phi$

PCs: $x = r\cos\theta \Rightarrow$ $\boxed{x = \rho\sin\phi\cos\theta}$

PCs: $y = r\sin\theta \Rightarrow$ $\boxed{y = \rho\sin\phi\sin\theta}$

$\cos\phi = \frac{z}{\rho} \Longrightarrow$ $\boxed{z = \rho\cos\phi}$

Why are you so sinful!

Cartesian Cs → SCs $(x,y,z) \to (\rho, \phi, \theta)$

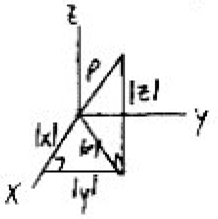
Find ρ

Idea: Use Distance Formula, Pyth. Thm., Trig IDs (Pyth. IDs).

Try them on SCs for x, y, z.

$\rho^2 = x^2+y^2+z^2$ $(\rho \geq 0)$

$$\boxed{\rho = \sqrt{x^2+y^2+z^2}}$$



$\rho^2 = r^2+z^2$
$= x^2+y^2+z^2$

Find ϕ

$z = \rho\cos\phi$

$\cos\phi = \frac{z}{\rho}$

$$\boxed{\phi = \cos^{-1}\left(\frac{z}{\rho}\right)}$$

Correct, because $\underbrace{0 \leq \phi \leq \pi}_{\text{Range of } \cos^{-1}}$

If $\rho = 0 \Rightarrow P$ is 0
$\Rightarrow \phi$ can be anything: $0 \leq \phi \leq \pi$

Find θ

PCs: $\tan\theta = \frac{y}{x}$ $(x \neq 0)$
Watch Quadrant!

there are ∞ many sol'ns. for θ in R

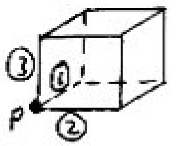
or Use $x = \rho\sin\phi\cos\theta$, or $y = \rho\sin\phi\sin\theta$

Known   Found

(D) Volume of a Spherical Box (Rough analysis!)

Rectangular/Cartesian box:

(1) Fix y, z; Vary x (here, $x \downarrow$)

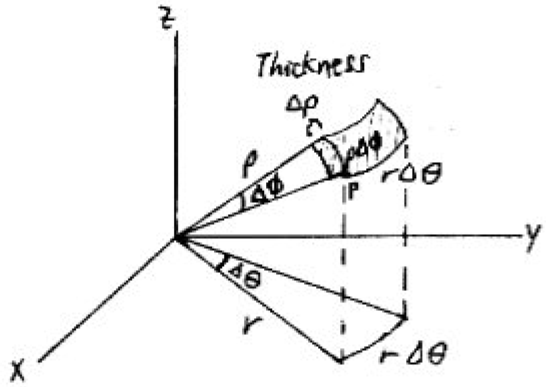


(2) Fix x, z; Vary y

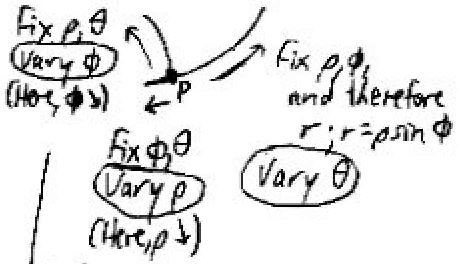
(3) Fix x, y; Vary z

$V = \Delta x \Delta y \Delta z$

ignore that we decreased x; don't worry about $|\Delta x|$



$\Delta\phi$: Think slots arm
$\Delta\theta$: Think roulette wheel



Still, let's say $\Delta\phi > 0$, $\Delta\rho > 0$.

If $\Delta\rho \approx 0$, $\Delta\phi \approx 0$, $\Delta\theta \approx 0 \Rightarrow$ This box $\approx$ Rectangular Box

Volume, $\Delta V \approx (r\Delta\theta)(\rho\Delta\phi)(\Delta\rho)$

SC: $\rho\sin\phi$

$$\Delta V \approx \rho^2 \sin\phi \, \Delta\rho \Delta\phi \Delta\theta$$

$$\boxed{dV = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta}$$

(You'll prove this in 17.9 HW.)

Ⓔ Old Ex (17.7, #33 : See 17.7.9)

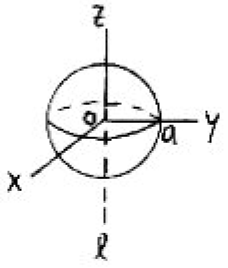
A spherical solid has radius a, and the density at $P(x,y,z)$ is directly proportional to the distance from P to a fixed line ℓ through the center of the solid. Find its mass.

Sol'n

Orient the Sphere "Q"

Let its center be at O.
Let the z-axis be ℓ.

Sketch Q (Optional?)



A box in SCs!

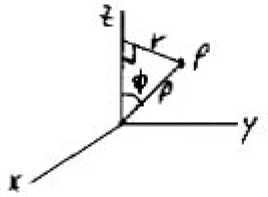
Find density, δ

$$\delta = k\sqrt{x^2+y^2} \quad \text{or} \quad kr$$

$x = \rho\sin\phi\cos\theta$

$y = \rho\sin\phi\sin\theta$

$r = \rho\sin\phi$

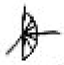


$$= k\rho\sin\phi$$

Find mass, m, of the solid

$$m = \iiint_Q \underbrace{k\rho\sin\phi}_{\delta \text{ func.}} \; \underbrace{dV}_{=\rho^2\sin\phi\, d\rho\, d\phi\, d\theta}$$

Fix θ:



really a half-plane

Then, fix ϕ: Pick a ray

Then, ρ: $0 \to a$

$$= k\int_{\theta=0}^{\theta=2\pi}\int_{\phi=0}^{\phi=\pi}\int_{\rho=0}^{\rho=a} \underbrace{\rho^3\sin^2\phi}\, d\rho\, d\phi\, d\theta$$

constants, separable $\Rightarrow$ We can separate $\int$s.

$$= k\underbrace{\left[\int_{\theta=0}^{\theta=2\pi} d\theta\right]}\underbrace{\left[\int_{\phi=0}^{\phi=\pi}\sin^2\phi\, d\phi\right]}\underbrace{\left[\int_{\rho=0}^{\rho=a}\rho^3\, d\rho\right]}$$

$= [\theta]_0^{2\pi}$
$= 2\pi$

Use a PRI.
$= \int_0^\pi \frac{1-\cos(2\phi)}{2}d\phi$
$= \frac{1}{2}\left[\phi - \frac{1}{2}\sin(2\phi)\right]_0^\pi$
$= \frac{1}{2}\left(\left[\pi - \frac{1}{2}\underbrace{\sin(2\pi)}_{=0}\right] - [0]\right)$
$= \frac{\pi}{2}$

$= \left[\frac{\rho^4}{4}\right]_0^a$
$= \frac{a^4}{4}$

$$= k(\cancel{2}\pi)\left(\frac{\pi}{\cancel{2}}\right)\left(\frac{a^4}{4}\right)$$

$$= \boxed{\frac{\pi^2 k a^4}{4}}$$

Same as before! Easier!

We didn't have complete separability before. Here, using SCs, all limits of $\int$ are constants.

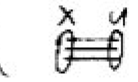
17.9: CHANGE OF VARIABLES and JACOBIANS

(A) Calc I: A New Look at u-Subs

Ex $\int_1^2 e^{3x}\,dx$

Let $u = 3x$ $\xrightarrow[\text{Inverse Idea}]{\text{Solve for } x}$ $x = \frac{1}{3}u$ ← We can think of the sub. this way: $x = f(u)$

($u(x)$: 1-1 func.


)

$du = 3\,dx$ $\qquad\qquad$ $dx = \frac{1}{3}\,du$

$\frac{du}{dx}$ $\qquad\qquad$ $\frac{dx}{du}$

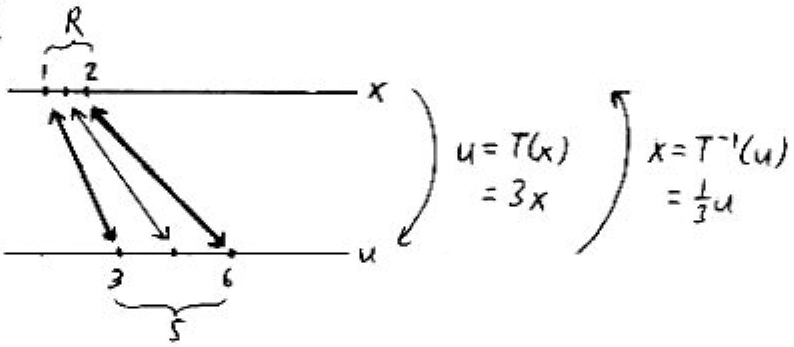
$$\left.\frac{dx}{du}\right|_u = \frac{1}{\left.\frac{du}{dx}\right|_x}$$

corresponding values

Change the limits of $\int$

$x = 1 \Longrightarrow u(1) = 3$
$x = 2 \Longrightarrow u(2) = 6$

Idea



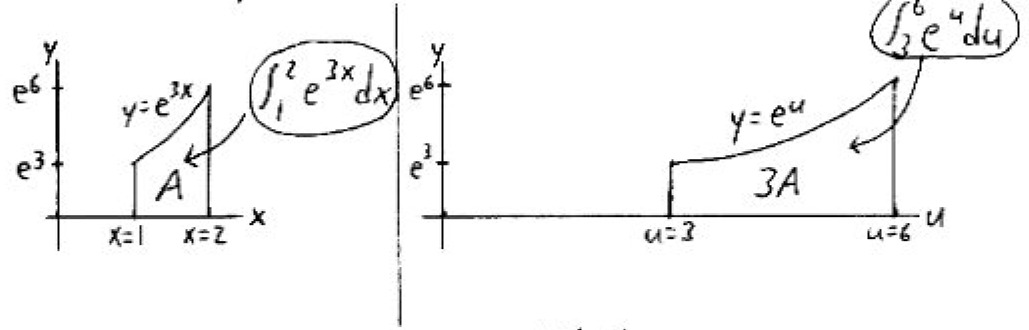
T is a 1-1 transformation of coordinates.
1-1 correspondence between x-values in R, u-values in S.

$$\int_1^2 e^{3x}\,dx = \int_{\underset{u(1)}{3}}^{\overset{u(2)}{6}} e^u \cdot \frac{1}{3}\,du$$

Why do we need $\left(\frac{dx}{du}\right)$?

Without it...

(Not to scale)



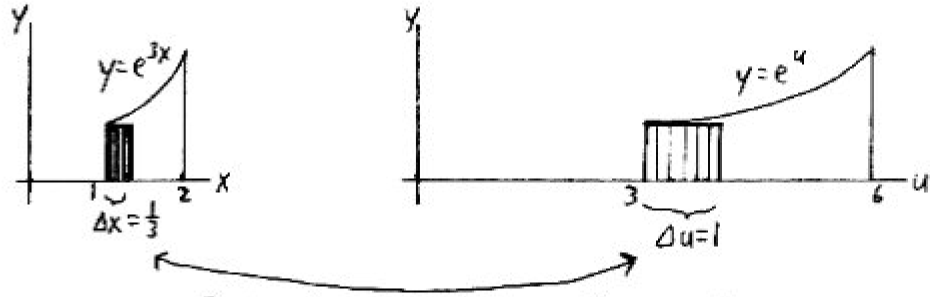
We'd get 3 times the area!! ☆
We need to <u>compensate</u> with a $\frac{1}{3}$.

<u>Why ☆?</u>

Think: Riemann rectangles

$du = 3dx$
$\Delta u = 3\Delta x$

If $\Delta x = \frac{1}{3} \Rightarrow \Delta u = 1$



Rectangles stretch along w/ horizontal axis →

$\rightarrow \quad \frac{du}{dx} = 3$

$= $ stretching factor $x \to u$ (Think: $\frac{du}{dx}$s)

$\longleftarrow \quad \frac{dx}{du} = \frac{1}{3}$

$=$ compensation factor

$=$ stretching factor $u \to x$ (compression, since $0 < \frac{1}{3} < 1$)

$$\int_1^2 e^{3x}\,dx = \int_3^6 e^u \cdot \frac{1}{3}\,du$$

Need $\frac{dx}{du}$

If you use $\left|\frac{dx}{du}\right|$, you can adopt the convention $\int_{\text{lower \#}}^{\text{upper \#}}$

Ex $\int_1^2 e^{-3x}\,dx = \int_{-3}^{-6} e^u \cdot \overbrace{\left(-\frac{1}{3}\right)}^{dx/du} du$

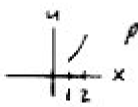
$= \int_{-6}^{-3} e^u \cdot \left|-\frac{1}{3}\right| du$

Idea: $\int_b^a \sim = -\int_a^b \sim$

Ex $\int_1^2 x e^{x^2}\,dx$

Let $u = x^2 \Longrightarrow x = \pm\sqrt{u}$ ($+$: x in $[1,2]$)

($u(x)$: 1-1 func. on $[1,2]$



passes HLT)

$dx = \underbrace{\frac{1}{2\sqrt{u}}}_{\frac{dx}{du}} du$

$du = 2x\,dx$ $\quad \frac{du}{dx}$ = instantaneous stretching factor. ☆

It changes as x changes.

Note:
$\frac{du}{dx} = 2x$
$\Rightarrow \frac{dx}{du} = \frac{1}{2x}$
$= \frac{1}{2\sqrt{u}}$

$$\int_1^2 x e^{x^2}\,dx = \int_{1 \leftarrow u(1)}^{4 \leftarrow u(2)} \sqrt{u}\, e^u \cdot \frac{1}{2\sqrt{u}}\,du$$

Need $\frac{dx}{du}$ as an instantaneous compensation factor.

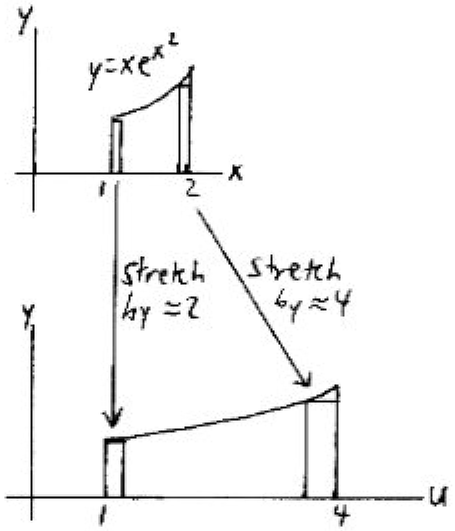
It changes as u changes.

Ⓐ Idea: Riemann rectangles

$du = 2x\,dx$
$\Delta u \approx 2x\,\Delta x$

If $x \approx 1$, then $\Delta u \approx 2\Delta x$.

If $x \approx 2$, then $\Delta u \approx 4\Delta x$.



Stretching factors change.

Compensation factors change.

Image: We're stretching the x-axis like a piece of taffy in which some parts are stretched more than others.
The corresponding rectangles are stretched in the same way.

Carl Jacobi (German, 1804-1851)

(B) Jacobians

are compensation factors for multiple $\int$s when we change variables to

(1) Simplify the region of integration, and/or

(2) Simplify the integrand.

(I) If $x = f(u)$,

Then, $dx = \left|\frac{dx}{du}\right| du$, if you always $\int_{\text{lower \#}}^{\text{higher \#}}$.

(II) If $\begin{cases} x = f(u,v), \\ y = g(u,v) \end{cases}$ have cont. 1^{st}-order PDs where we care

Hard proof: Larson 6ed p. 476

Then, $dA = \left| \begin{array}{c c} & \begin{array}{cc} x & y \end{array} \\ \begin{array}{c} u \\ v \end{array} & \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} \end{array} \right| du\, dv$

abs. value

determinant = the Jacobian of x and y wrt u and v

$=$ " $\dfrac{\partial(x,y)}{\partial(u,v)}$ " ← x,y on top

We require that this is never 0 where we care.

what would it compensate for?

so it can't change sign (assume 1^{st}-order PDs cont.)

Stretches/compresses a 2-D region, (corresponding 3-D solid)

Note $\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix}$

OK to switch: $|\text{matrix } A| = |A^T|$

T: transpose: switch rows, cols

Key: $\frac{\partial}{\partial}$ ← x, y on top; ← u, v on bottom

(∫∫∫) If $\begin{cases} x = f(u,v,w) \\ y = g(u,v,w) \\ z = h(u,v,w) \end{cases}$

$$\text{Then, } dV = \underbrace{\left| \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \underbrace{\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \\ \frac{\partial x}{\partial w} & \frac{\partial y}{\partial w} & \frac{\partial z}{\partial w} \end{vmatrix}}_{\text{"}\frac{\partial(x,y,z)}{\partial(u,v,w)}\text{"}} \end{array} \begin{array}{c} \\ \leftarrow u \\ \leftarrow v \\ \leftarrow w \end{array} \right|} \, du\, dv\, dw$$

How to Ace

If $\vec{r} = \langle x, y, z \rangle$, then this $= \left\| \begin{matrix} \leftarrow \frac{\partial \vec{r}}{\partial u} \rightarrow \\ \leftarrow \frac{\partial \vec{r}}{\partial v} \rightarrow \\ \leftarrow \frac{\partial \vec{r}}{\partial w} \rightarrow \end{matrix} \right\|$

$= \left| \text{TSP of } \frac{\partial \vec{r}}{\partial u}, \frac{\partial \vec{r}}{\partial v}, \frac{\partial \vec{r}}{\partial w} \right|$

= Volume of parallelepiped determined by

Stretches/compresses a 3-D region.

Ⓒ A New Look at PCs

Ex A region R in the xy-plane consists of points (x,y):

$$x = \underbrace{r\cos\theta}_{f(r,\theta)}$$

$$y = \underbrace{r\sin\theta}_{g(r,\theta)}$$

We're expressing the old vars. in terms of the new vars.

This turns out to be easier for us!

where $1 \le r \le 2$, $\frac{\pi}{6} \le \theta \le \frac{\pi}{3}$ } determine a region S in the $r\theta$-plane

What is dA?

$$dA = \left|\begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} \end{vmatrix}\right| dr\, d\theta$$

$$= \left|\begin{vmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{vmatrix}\right| dr\, d\theta$$

$$= |r\cos^2\theta + r\sin^2\theta|\, dr\, d\theta$$

$$= |r\underbrace{(\cos^2\theta + \sin^2\theta)}_{=1}|\, dr\, d\theta$$

$$= |r|\, dr\, d\theta$$

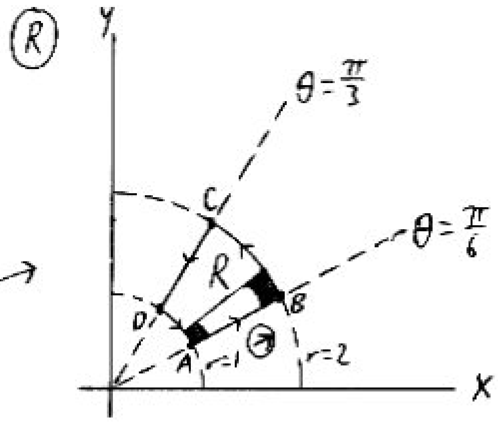
Given: $1 \le r \le 2$

$$= r\, dr\, d\theta$$ ☺

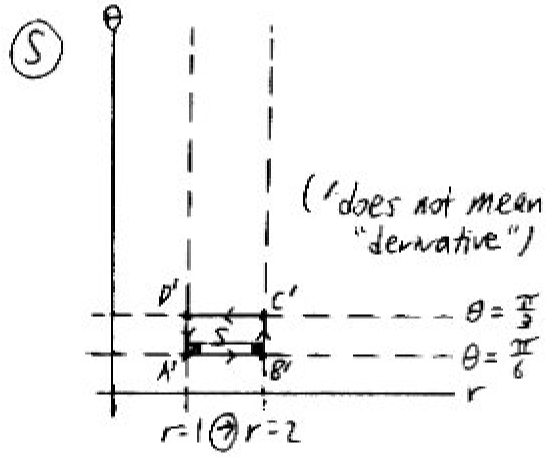
$r\,dr\,d\theta$

$\Rightarrow$ Greater stretching of areas as $r \to 2$ (Really compensating for)

Graph R, S



Arrows indicate positive orientation: R always on the left



(′ does not mean "derivative")

These are r-curves ($r = \#$), θ-curves ($\theta = \#$) in the xy-plane.

Think: Level curves.

Jacobian turns out to be positive in value, so positive orientation is retained.

Note 1: We like that the boundaries of R, S are piecewise smooth, simple, closed, bounded curves.

- piecewise: has pieces parametrizable by param'ns where derivs. cont., $\neq \vec{0}$ except maybe at endpts.
- simple: no @
- closed: no C...
- bounded: no ∞

Note 2: We like that, as we trace the boundary of R once in one direction, we trace the boundary of S once in one direction.

Note 3: Had the Jacobian been negative in value, the orientation would have been reversed along the S-boundary.

Note 4: We like that S is simpler than R. The change of variables may help!

See Larson 6ed, pp. 975-8

(1) Ex

Let R be the region bounded by:

$$\begin{aligned} x - 2y &= 0 \\ x - 2y &= 2 \\ x + y &= 1 \\ x + y &= 3 \end{aligned}$$

Evaluate $\iint_R (x+y)\sin(x-2y)\, dA$.

Sol'n

(1) Change of variables

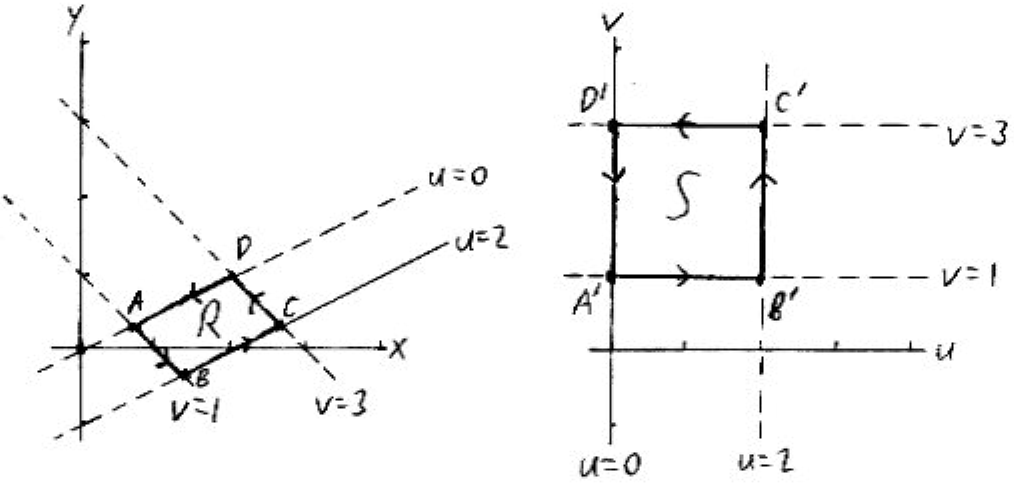
Let $\begin{cases} u = x - 2y \\ v = x + y \end{cases}$ or

(2) New limits of $\iint$

$u=0$
$u=2$
$v=1$
$v=3$

(2)

Note: What do R and S look like?



Jacobian turns out to be positive in value, so positive orientation is retained.

③ Solve for x,y in terms of u,v ((an skip if do ④ next page)

$$u = x - 2y$$
$$v = x + y \quad \leftarrow \cdot(-1)$$

$$\begin{aligned} u &= x - 2y \\ -v &= -x - y \\ \hline u - v &= -3y \end{aligned} \quad \downarrow \text{Add}$$

$$y = -\tfrac{1}{3}(u-v) \text{ or } \tfrac{1}{3}v - \tfrac{1}{3}u$$

Find x

$$v = x + y$$
$$v = x - \tfrac{1}{3}(u-v)$$
$$x = v + \tfrac{1}{3}(u-v)$$
$$x = \tfrac{1}{3}u + \tfrac{2}{3}v \text{ or } \tfrac{1}{3}(u+2v)$$

Another Method (because of linearity)

$$\begin{bmatrix} u \\ v \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix}}_{A} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} u \\ v \end{bmatrix}$$

switched flipped signs

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{\underset{\det(A)}{3}} \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

$$\begin{cases} x = \tfrac{1}{3}(u+2v) \\ y = \tfrac{1}{3}(-u+v) \end{cases}$$

Note If we had done something dumb, like

$$\begin{cases} u = x - 2y \\ v = x - 2y \end{cases},$$

then $\det\left(\begin{bmatrix} 1 & -2 \\ 1 & -2 \end{bmatrix}\right) = 0.$

u=0, v=0

u=2, v=2

④ Find Jacobian

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$x = \tfrac{1}{3}u + \tfrac{2}{3}v$$
$$y = -\tfrac{1}{3}u + \tfrac{1}{3}v$$

$$= \begin{vmatrix} \frac{1}{3} & -\frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} \end{vmatrix}$$

$$= \frac{1}{9} + \frac{2}{9}$$

$$= \frac{1}{3}$$

or ④* (Can skip ③)

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$u = x - 2y$$
$$v = x + y$$

$$= \begin{vmatrix} 1 & -2 \\ 1 & 1 \end{vmatrix}$$

$$= 3$$ ← If this had x and/or y in it, express in terms of u,v

$$\Rightarrow \frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{\frac{\partial(u,v)}{\partial(x,y)}}$$

$$= \frac{1}{3}$$

Note

$$\begin{tikzcd} R \arrow[r, bend left, "\text{Stretching factor: } 3"] & S \arrow[l, bend left, "\text{Compensating factor: } \frac{1}{3}"] \end{tikzcd}$$

$$\text{Area}(S) = 3 \cdot \text{Area}(R)$$

⑤ Set up ∬, and Evaluate

$$\iint_R \underbrace{(x+y)\sin(x-2y)}_{\text{cont. on } R}\, dA$$

$$= \iint_S v \sin u \cdot \underbrace{\left|\frac{\partial(x,y)}{\partial(u,v)}\right|}_{=\frac{1}{3}}\, dv\,du$$

or

$$= \frac{1}{3}\int_{u=0}^{u=2}\int_{v=1}^{v=3} v \sin u \; dv\,du$$

(Remember: $\int_{\text{lower \#}}^{\text{higher \#}}$)

$$= \frac{1}{3}\underbrace{\int_0^2 \sin u\, du}\ \underbrace{\int_1^3 v\, dv}$$

$$= [-\cos u]_0^2 \qquad = \left[\frac{v^2}{2}\right]_1^3$$

$$= -\cos 2 + \underbrace{\cos 0}_{=1} \qquad = \frac{9}{2} - \frac{1}{2}$$

$$= 1 - \cos 2 \qquad = 4$$

$$= \frac{1}{3}(1-\cos 2)(4)$$

$$= \boxed{\frac{4(1-\cos 2)}{3}}$$

$$\approx 1.8882$$

<u>Ex</u> (#20)

$$\iint_R (3x-4y)\,dx\,dy$$

Boundary of R: $y=3x$, $y=\frac{1}{2}x$, $x=4$

Change of vars.: $x=u-2v$
$y=3u-v$

<u>Sol'n</u>

Rewrite the Integrand:

$$3x-4y = 3(u-2v)-4(3u-v)$$
$$= -9u-2v$$

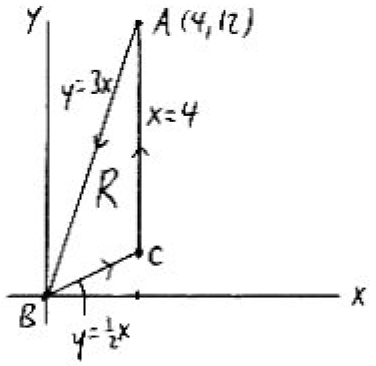
Jacobian:

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & -2 \\ 3 & -1 \end{vmatrix}$$

$$= 5$$

What's R?



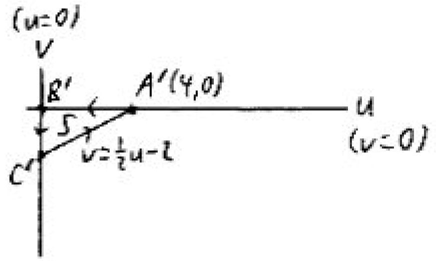
What's S?

Boundaries:

$$y = 3x \qquad y = \tfrac{1}{2}x \qquad x = 4$$

$$(3u - v) = 3(u - 2v) \qquad (3u - v) = \tfrac{1}{2}(u - 2v) \qquad (u - 2v) = 4$$

$$v = 0 \qquad u = 0 \qquad v = \tfrac{1}{2}u - 2$$



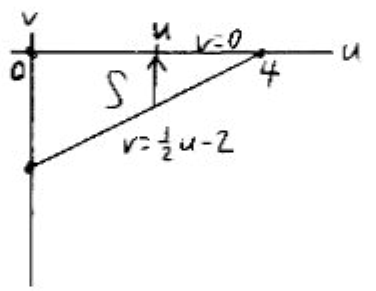
Note: $A(\overset{x}{4}, \overset{y}{12}) \Rightarrow A'(\overset{u}{4}, \overset{v}{0})$

← from this graph, or

Solve: $\begin{cases} 4 = u - 2v \\ 12 = 3u - v \end{cases}$

$\Rightarrow (u, v) = (4, 0)$

Close-Up:



Note: $\text{Area}(R) = 5 \cdot \text{Area}(S)$

$\text{Area}(S) = \frac{1}{5} \cdot \text{Area}(R)$

17.9.14

$$\iint_R (3x-4y)\,dx\,dy = \iint_S (-9u-2v)\cdot \underbrace{\left|\frac{\partial(x,y)}{\partial(u,v)}\right|}_{=(5)} \underbrace{du\,dv}_{\text{or } dv\,du}$$

$$= 5\iint_S (-9u-2v)\,dv\,du$$

Setup →

$$= 5\int_{u=0}^{u=4}\int_{v=\frac{1}{2}u-2}^{v=0} (-9u-2v)\,dv\,du$$

↑Remember, $\int_{\text{lower value}}^{\text{higher value}}$

No more unusual than $\int_{-3}^{0} \sim$.

⋮

$$= \boxed{-\frac{640}{3} \quad (= -213.\overline{3})}$$

JUMBLING TSPs

How is $(\mathbf{a}\times\mathbf{b})\bullet\mathbf{c}$ related to $(\mathbf{a}\times\mathbf{c})\bullet\mathbf{b}$?

Determinant approach:

$$(\mathbf{a}\times\mathbf{b})\bullet\mathbf{c} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$(\mathbf{a}\times\mathbf{c})\bullet\mathbf{b} = \begin{vmatrix} a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

The determinant forms differ only by a single switch of two rows, so they differ only by a sign.

$$\boxed{(\mathbf{a}\times\mathbf{b})\bullet\mathbf{c} = -\left[(\mathbf{a}\times\mathbf{c})\bullet\mathbf{b}\right]}$$

Geometric approach:

Both $\left|(\mathbf{a}\times\mathbf{b})\bullet\mathbf{c}\right|$ and $\left|(\mathbf{a}\times\mathbf{c})\bullet\mathbf{b}\right|$ represent the volume of the parallelepiped determined by the position vectors for **a**, **b**, and **c**. This is consistent with the box above.

How is $(\mathbf{a}\times\mathbf{b})\bullet\mathbf{c}$ related to $(\mathbf{b}\times\mathbf{c})\bullet\mathbf{a}$?

Determinant approach:

$$(\mathbf{a}\times\mathbf{b})\bullet\mathbf{c}=\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$(\mathbf{b}\times\mathbf{c})\bullet\mathbf{a}=\begin{vmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \end{vmatrix}$$

The determinant forms differ by <u>two</u> switches of pairs of rows, so there is a "double negative" effect, and the determinants are equal.

$$\boxed{(\mathbf{a}\times\mathbf{b})\bullet\mathbf{c}=(\mathbf{b}\times\mathbf{c})\bullet\mathbf{a}}$$

Geometric approach:

(Similar to the first example)

Remember that dot products are commutative, and cross products are anticommutative, so it may be easy to relate some jumbles. For example,

$$\begin{aligned}(\mathbf{a}\times\mathbf{b})\bullet\mathbf{c}&=\mathbf{c}\bullet(\mathbf{a}\times\mathbf{b})\\ &=-\left[\mathbf{c}\bullet(\mathbf{b}\times\mathbf{a})\right]\end{aligned}$$

Basically, if you perform a TSP jumble of three vectors in V_3 that makes sense (you only take dot products of two vectors of the same length, and you only take cross products of two vectors in V_3), you either get the original TSP value or its opposite. This is consistent with the geometric interpretation of the absolute value of a TSP as the volume of a parallelepiped determined by the position vectors for the three constituent vectors.

REVIEW: CH.17

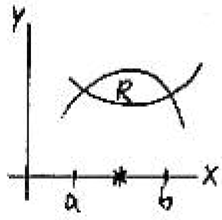
17.1/17.2: $\iint$s

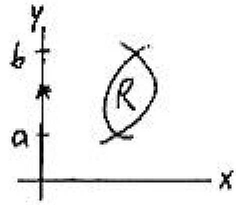
$\iint_R 1\, dA = \text{Area of } R$

$dA = dx\,dy$ or $dy\,dx$

① Sketch R

Solve systems to locate intersection points.





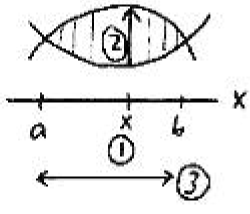
Identify which graph is on top vs. bottom
or right vs. left.

Use test values? *

If  ⇒ Split R.

② Set up limits of $\iint$

Ex



① Aim outer var

can't depend on

$$\int_{x=a}^{x=b} \int_{y=bot}^{y=top} 1 \, dy \, dx$$

② Shoot inner var

③ Slide/Scan along outer var

Reversing Order of $\iint$

Why? If R-scan is awkward, or
if hard to do inner $\int$.

If $\int_{x=a}^{x=b} \int_{y=c}^{y=d} \sim dy\,dx$

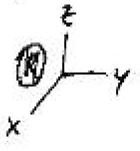
Can if all limits are constant

Otherwise, Sketch R
Switch scan (slide)/outer/aim variable
in analysis.

$\iint_R \underbrace{f(x,y)}_{\text{if } \geq 0 \text{ on } R} dA$ = Volume between f graph, xy-plane over R.

$\iint_R [\underset{\text{top}}{f(x,y)} - \underset{\text{bottom}}{g(x,y)}]\, dA$ = Volume between f, g graphs "over" R.
(below OK)

Can shift perspective



$dA = dx\,dz$

Polar

17.3: PCs

$$\iint_R f(x, y)\, dA$$

R ← Sketch!

x ← $r\cos\theta$

y ← $r\sin\theta$

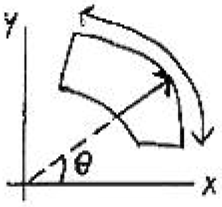
dA → $r\,dr\,d\theta$ or

$$x^2+y^2=r^2$$

Ex $dA = r\,dr\,d\theta$

Shoot (dr), Fix ($d\theta$)

Slide



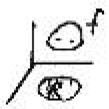
Why use PCs?

If R is bounded by lines/rays, circular arcs, basic polar graphs, or

If the integrand has x^2+y^2, etc.

17.4: SURFACE AREA

$$S = \iint_R \sqrt{1 + [f_x(x,y)]^2 + [f_y(x,y)]^2}\, dA$$



Use PCs?

17.5: $\iiint$s

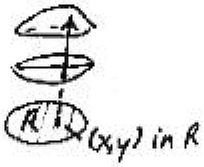
$$\iiint_Q f(x,y,z)\, \underbrace{dV}_{= dx\,dy\,dz \text{ (or permutation)}}$$

Ex

$$\iint_R \left[\int_{?}^{??} f(x,y,z)\, dz \right] dx\, dy$$

or

How do we shoot z if we fix x, y?



What about R?

Can you find the projection of Q on a coord. plane? (Here, xy-plane.)

Maybe look for 2 cylinders that "trap" space. Let R be bounded by their traces in a coord. plane they're $\perp$ to.

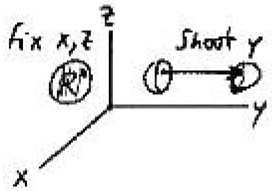
Use symmetry? PCs?

consider integrand, inner $\int$s

Can shift perspective

$$\iint_R \left[\int_{y=?}^{y=??} f(x,y,z)\, dy \right] dx\, dz$$

or



17.6: CENTER OF MASS

Mass = m

$$= \iint_R \underbrace{\delta(x,y)}_{\text{density}} \, dA$$

constant for a homogeneous lamina

Center of mass = $(\bar{x}, \bar{y})$

$$\bar{x} = \frac{\iint_R x \, \delta(x,y) \, dA}{m} \quad , \quad \bar{y} = \text{'' } y \text{ '' ''}$$

Can use symmetry about $x=0$ (yz-plane), say, if

δ is even in x, and
R is sym. about $x=0$.

$m = 2 \iint_{\text{new } R}$, or $\bar{x} = 0$

Moments of inertia

$$I_y = \iint_R x^2 \, \delta(x,y) \, dA$$

$$I_x = \text{'' } y^2 \text{ '' ''}$$

$$I_0 = \text{'' } r^2 \text{ '' ''}$$

3D similar, except

$$I_z = \iiint_Q \underbrace{(x^2+y^2)}_{\text{sq. dist. from } z\text{-axis}} \delta(x,y,z) \, dV \text{, etc.}$$

Cylindrical

17.7: CYL. Cs

(r, θ, z)

$\underbrace{r, \theta}_{\text{Polar}}$, z ← Cartesian

Basic graphs

$dV = r\,dr\,d\theta\,dz$

often

$$\iint_R \left[\int_{z=?}^{z=?!} \sim dz \right] r\,dr\,d\theta$$

or

Sketch!

Think: Polar

Spherical

17.8: SCs

$[0,\infty)$ $[0,\pi]$

(ρ, ϕ, θ)

Basic graphs

Sketch → Formulas

$r = \rho \sin\phi$

$x = r\cos\theta = \rho\sin\phi\cos\theta$

$y = r\sin\theta = \rho\sin\phi\sin\theta$ y so sinful?

$z = \rho\cos\phi$

$\rho = \sqrt{x^2+y^2+z^2}$

$\phi = \cos^{-1}\left(\frac{z}{\rho}\right)$

$\tan\theta = \frac{y}{x}$, watch Q!

$dV = \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$

17.9: JACOBIANS, CHANGE OF VARIABLES

$$dA = \left\| \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} \right\| du\,dv \quad , \text{where} \quad \left.\begin{array}{l} x = f(u,v), \\ y = g(u,v) \end{array}\right\} \begin{array}{l} \text{1-1} \\ (u,v) \leftrightarrow (x,y) \end{array}$$

$$dV = \left\| \begin{array}{c} \text{similar,} \\ 3\times 3 \end{array} \right\| du\,dv\,dw$$

$$\left.\begin{array}{l} \text{Need new limits for } \iint, \iiint \\ \to u,v,w \text{ for integrand} \end{array}\right\} \begin{array}{l} \text{Influence} \\ \text{choice} \\ \text{of sub} \end{array}$$