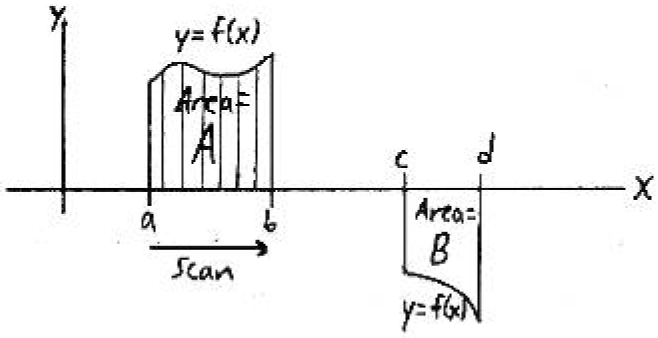
# CH. 17: MULTIPLE INTEGRALS



## 17.1: DOUBLE INTEGRALS ($\iint$), and 17.2: AREA and VOLUME

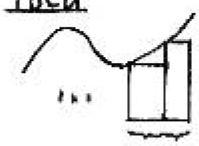
### Ⓐ Intro

**Calc I**




$$\boxed{\int_a^b f(x)\,dx = A} \qquad \boxed{\int_c^d f(x)\,dx = -B}$$

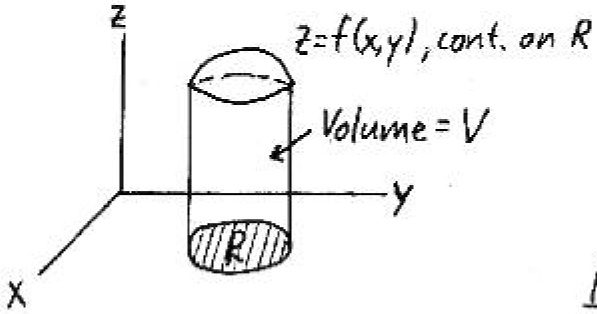
**Idea**




Add areas

Widest width → 0
(⇒ # rectangles → ∞)

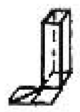
**Now**




$$\boxed{\iint_R f(x,y)\,dA = V}$$

Below xy-plane ⇒ −V

**Idea**

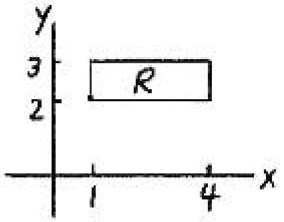


Add volumes

Longest diagonal → 0
(⇒ # boxes → ∞)

## (B) Rectangular R

Ex R is given by $\{(x,y) \mid 1 \le x \le 4 \text{ and } 2 \le y \le 3\}$.




Find the volume between the graphs of

$$\underbrace{z}_{f(x,y)} = x^2y + y^2 \text{ and } \underbrace{z=0}_{xy\text{-plane}} \text{ over } R.$$

Sol'n

Note $x^2y + y^2 \ge 0$ on $R$, so our solid lies entirely $\underset{\text{(and on)}}{\text{above}}$ the xy-plane.

Guido Fubini (1879-1943)

By Fubini's Theorem,

$$\iint_R f(x,y)\,dA =$$

#1 $\int_2^3 \int_1^4 \underbrace{(x^2y + y^2)}\,dx\,dy$

Easier to do partial $\int$ wrt x, so do it 1st

Then, $\int$ wrt y

Inside-out

or #2 $\underbrace{\int_1^4 \underbrace{\int_2^3 (x^2y + y^2)\,dy}\,dx}$

(#1) $\int_2^3 \left[ \int_1^4 (x^2 \underbrace{y} + \underbrace{y^2})\, dx \right] dy$

Treat $y$ as a constant.

$= \int_2^3 \left[ \left(\frac{x^3}{3}\right) y + y^2 x \right]_{x=1}^{x=4} dy$

↑ Optional reminder

Think: $\int x^2 \cdot 7\, dx = \left(\frac{x^3}{3}\right) \cdot 7 + C$

$\int 7\, dx = 7x + C$

$= \int_2^3 \left( \left[ \frac{(4)^3}{3} y + y^2 (4) \right] - \left[ \frac{(1)^3}{3} y + y^2 (1) \right] \right) dy$

$= \int_2^3 \left( \frac{64}{3} y + 4y^2 - \frac{1}{3} y - y^2 \right) dy$

$= \int_2^3 (3y^2 + 21y)\, dy$ (Calc I!)

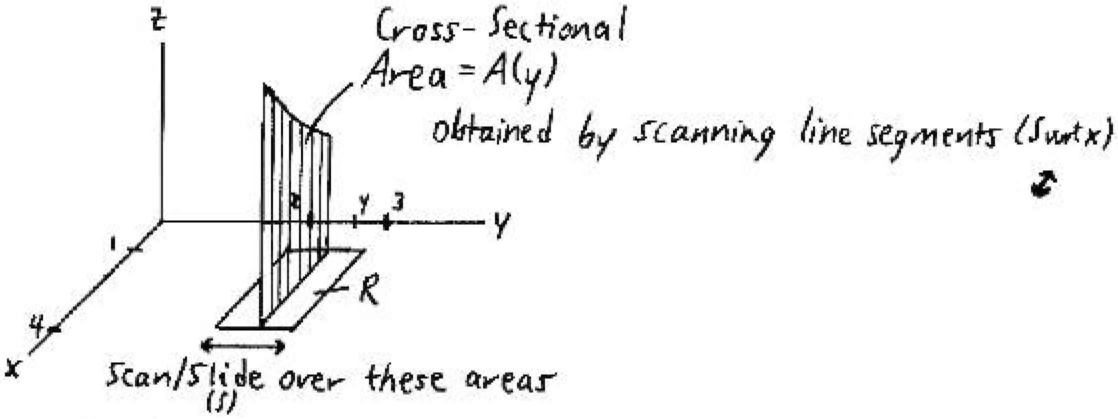
$= \left[ y^3 + 21 \left( \frac{y^2}{2} \right) \right]_2^3$

$= \left[ (3)^3 + 21 \cdot \frac{(3)^2}{2} \right] - \left[ (2)^3 + 21 \cdot \frac{(2)^2}{2} \right]$

Can do 13

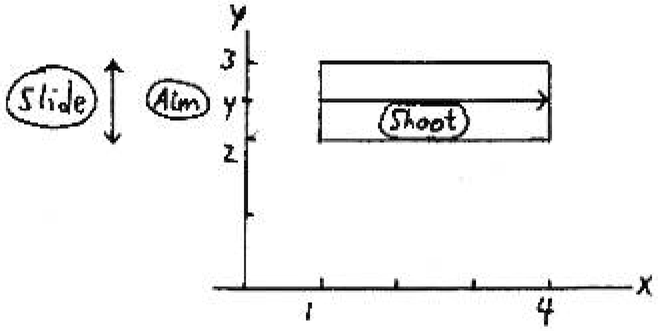
$= \boxed{\frac{143}{2} \text{ or } 71.5 \text{ cubic units}}$

Idea




$$\int_2^3 \underbrace{\underbrace{\int_1^4 (x^2 y + y^2)\, dx}_{\substack{= A(y) \\ \text{for fixed } y \text{ bet. } 2,3}}\ dy}_{\text{Slide (S wrt y)}}$$

## How did we scan over R?




Aim: Fix a y-value between 2, 3.
y is the outer variable.

Shoot: From $x=1$ to $x=4$.
x is the inner variable.

Slide: ($\int$) From $y=2$ to $y=3$.
outer

y: final scanning direction ↕

## Ⓒ Other Rs

Variation on 17.2, #32 (no "2")

**Ex** Find the volume, $V$, of the solid bounded by the graphs of:

You figure out

$$\left.\begin{array}{l} y = 2x^3 \\ y = x^4 \end{array}\right\}$$ (Determine $R$ in $xy$-plane)

We'll see why

$z - x - y = 4 \Leftrightarrow z = x + y + 4$ (Top surface) (Plane)

$z = 0$   $xy$-plane (Bottom surface)

### ① What is R?

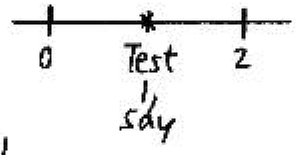
#### ①a Find any Intersection Points (Just "x" coords.? "y"?)

Solve $\begin{cases} y = 2x^3 \\ y = x^4 \end{cases}$

$$\begin{aligned} \Rightarrow 2x^3 &= x^4 \\ 0 &= x^4 - 2x^3 \\ 0 &= \underbrace{x^3}_{\Downarrow\ 0}\underbrace{(x-2)}_{\Downarrow\ 2} \end{aligned}$$

$x = 0, 2$

## (1b) Which graph is on top? bottom?



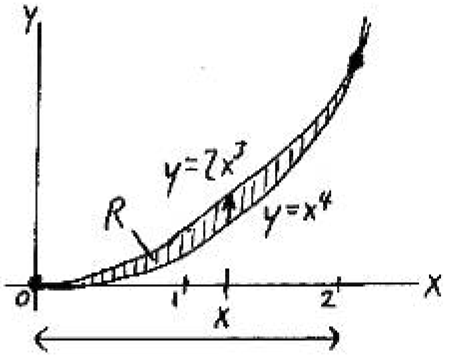

At $x=1$,

$$2x^3 = 2(1)^3 = 2 \quad \text{top}$$
$$x^4 = (1)^4 = 1 \quad \text{bottom}$$

$2>1$, so

where $0<x<2$

## (1c) Sketch R (maybe - I may want this on a test)




## (2) Set up ∬

### Idea

**Aim** Fix $x$ $(0 \le x \le 2)$
outer variable

**Shoot** ↑ From $y=x^4$ (bottom) to $y=2x^3$ (top) $y$: inner variable

**Slide** ↔ From $x=0$ to $x=2$
outer variable is overall "scan" variable

## Surfaces

On $R$, $z = x + y + 4 > 0$ top

$x, y \geq 0$

$z = 0$ bottom

$$V = \iint_R \Big[ \underbrace{(x+y+4)}_{\text{top } z} - \underbrace{(0)}_{\text{bottom } z} \Big] \, dA$$

**Note** It's OK if "bottom $z$" is sometimes negative on $R$, as long as it's below or at "top $z$."

$$= \int_{x=0}^{x=2} \int_{y=x^4}^{y=2x^3} (x+y+4) \, dy \, dx$$

must be constant relative to $x, y$

can depend on $x$ but not $y$ if $\int$ wrt $y$

Shoot

Slide

"$x =$" and "$y =$" optional but help!

③ Evaluate $\iint$ (I may or may not ask for this.)

$$V = \int_{x=0}^{x=2} \left[ \int_{y=x^4}^{y=2x^3} (x+y+4)\, dy \right] dx$$

$$= \int_{x=0}^{x=2} \left[ xy + \tfrac{1}{2}y^2 + 4y \right]_{y=x^4}^{y=2x^3} dx$$

$$= \int_{x=0}^{x=2} \Big( \left[ x(2x^3) + \tfrac{1}{2}(2x^3)^2 + 4(2x^3) \right] - \left[ x(x^4) + \tfrac{1}{2}(x^4)^2 + 4(x^4) \right] \Big) dx$$

$$= \int_0^2 \left( 2x^4 + \tfrac{1}{2}(4x^6) + 8x^3 - \left[ x^5 + \tfrac{1}{2}x^8 + 4x^4 \right] \right) dx$$

You're in danger of forgetting

$$= \int_0^2 \left( 2x^4 + 2x^6 + 8x^3 - x^5 - \tfrac{1}{2}x^8 - 4x^4 \right) dx$$

$$= \int_0^2 \left( -\tfrac{1}{2}x^8 + 2x^6 - x^5 - 2x^4 + 8x^3 \right) dx$$

$$= \left[ -\tfrac{1}{2}\left(\tfrac{x^9}{9}\right) + 2\left(\tfrac{x^7}{7}\right) - \tfrac{x^6}{6} - 2\left(\tfrac{x^5}{5}\right) + \overset{2}{\cancel{8}}\left(\tfrac{x^4}{\cancel{4}}\right) \right]_0^2$$

$$= \left[ -\tfrac{1}{2}\left(\tfrac{2^9}{9}\right) + 2\left(\tfrac{2^7}{7}\right) - \tfrac{2^6}{6} - 2\left(\tfrac{2^5}{5}\right) + 2(2)^4 \right] - [0]$$

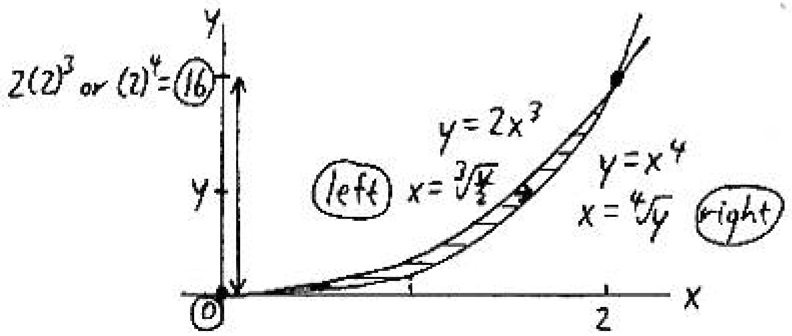
$$= \boxed{\frac{5248}{315}}$$

## (D) Reversing the Order of Integration

Same Ex $\int_{x=0}^{x=2}\int_{y=x^4}^{y=2x^3} (x+y+4)\,dy\,dx \Rightarrow \iint_R (x+y+4)\,dx\,dy$

Don't just switch!

Sketch R




Solve for x

$$y = 2x^3 \qquad\qquad y = x^4$$

$$\frac{y}{2} = x^3 \qquad\qquad x = \cancel{\pm}\sqrt[4]{y}$$

x in [0,2]

$$x = \sqrt[3]{\frac{y}{2}}$$

Aim Fix $y$ $(0 \le y \le 16)$
outer

Shoot → From $x=\sqrt[3]{\frac{y}{2}}$ (left) to $x=\sqrt[4]{y}$ (right)

Slide ↕ From $y=0$ to $y=16$

All but 17.2, #1

$$\boxed{\int_{y=0}^{y=16}\int_{x=\sqrt[3]{y/2}}^{x=\sqrt[4]{y}} (x+y+4)\,dx\,dy}$$

It may be easier (or necessary) to reverse the order because of $R$ and/or the form of $\underbrace{f(x,y)}$.

↑

R is vertically simple →  $\leftrightarrow x$ $\iint \sim dy\,\underset{\substack{\uparrow \\ \text{outer/} \\ \text{scan}}}{dx}$

Is it easier to $\int$ wrt $x$ or $\int$ wrt $y$ first?

horiz. → $\updownarrow y$  $\iint \sim dx\,dy$

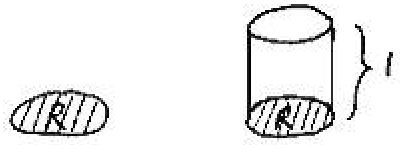
<u>Ex</u> (17.1.*46) $\int_0^9 \int_{\sqrt{y}}^3 \sin x^3 \, dx \, dy$

<u>BUT</u> I can't do $\int \sin x^3 \, dx$ (without series)

<u>BETTER</u> $\int \underbrace{\sin x^3}_{\text{"\#"}} \, dy = y \sin x^3$

17.1.11

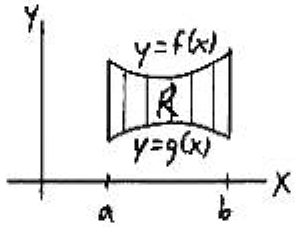
## (E) Area of R




Area = Volume (numerically)

Choose $f(x,y)=1$.

Ex




$$A = \iint_R 1 \, dA$$

$$= \int_{x=a}^{x=b} \int_{y=g(x)}^{y=f(x)} 1 \, dy \, dx$$

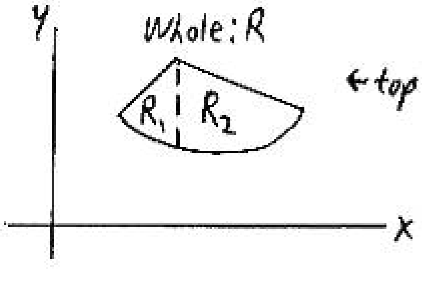
$$= \int_{x=a}^{x=b} \left[ y \right]_{y=g(x)}^{y=f(x)} dx$$

$$= \int_a^b [\underbrace{f(x)}_{\text{top}} - \underbrace{g(x)}_{\text{bottom}}] \, dx$$

Calc I! (6.1)

## (F) Splitting R

p.888




← top changes!

"$R = R_1 \cup R_2$"

union

$$\iint_R f(x,y)\,dA = \iint_{R_1} f(x,y)\,dA + \iint_{R_2} f(x,y)\,dA$$