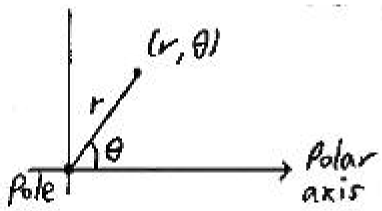
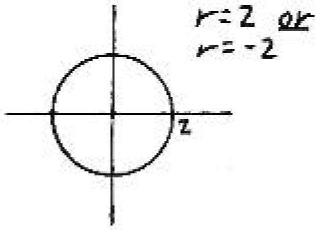
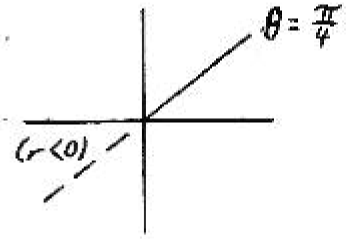
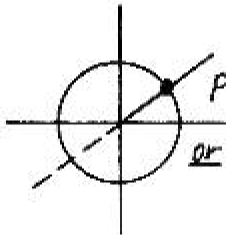
17.3: $\iint$s IN POLAR COORDS. (PCs)

(A) Review PCs







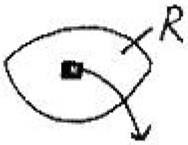


$P\left(2, \frac{\pi}{4} + 2\pi n\right)$ for any integer n

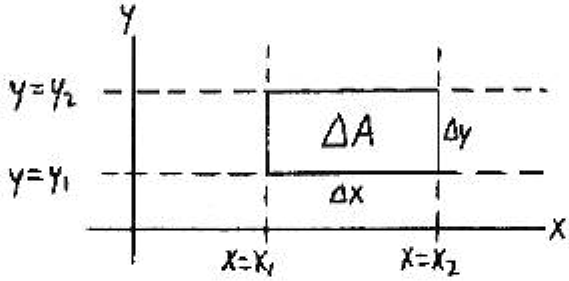
or $P\left(-2, \frac{5\pi}{4} + 2\pi n\right)$

Ⓑ Area of a Polar Rectangle

Ⓑ1 17.1/17.2: Cartesian Rectangle



(break up into rectangles for a Riemann sum)



In Riemann Sum,

$$\Delta A = \Delta x \, \Delta y$$

In $\iint$,

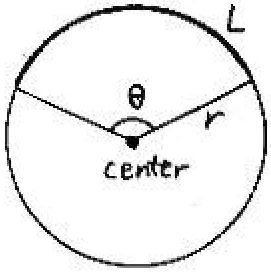
$$dA = dx \, dy$$

or

$$dA = dy \, dx$$

(B2) Arc Length "L" along a Circle

Measure angles in radians.

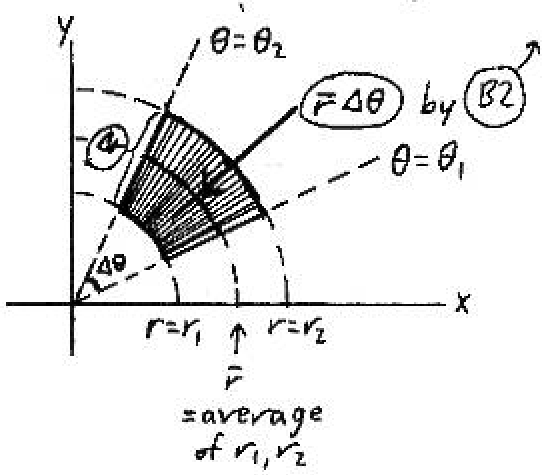


$L =$ (fraction of circle)(circumference)

$$L = \left(\frac{\theta}{\cancel{2\pi}}\right)(\cancel{2\pi} r)$$

$$\boxed{L = r\theta}$$

(B3) Area of a Polar Rectangle



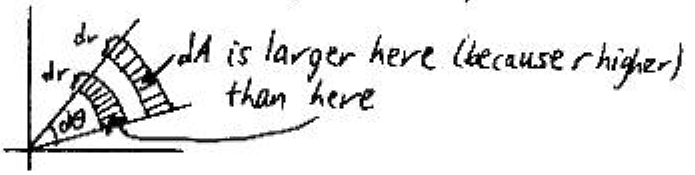
Think: fan/ windshield wiper (How to Ace)

Turns out (see (B4))

$$\text{Shaded } \Delta A = (\Delta r)(\bar{r}\,\Delta\theta)$$
$$= \bar{r}\,\Delta r\,\Delta\theta$$

In $\iint$, $\boxed{dA = r\,dr\,d\theta}$

Don't forget!! Idea: If $d\theta$, dr fixed,



Larson 6ed
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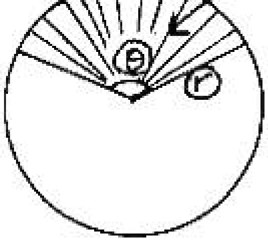
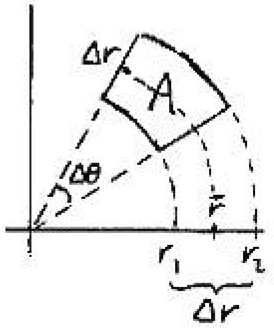
(B4) Optional: Why does $\Delta A = (\Delta r)(\bar{r}\Delta\theta) = \bar{r}\Delta r\Delta\theta$?



Sector Area = (fraction of circle)(area of circle)

$$= \left(\frac{\theta}{2\pi}\right)(\pi r^2)$$

$$= \frac{1}{2}r^2\theta$$



$$A = \frac{1}{2}r_2^2\Delta\theta - \frac{1}{2}r_1^2\Delta\theta$$

$$= \frac{1}{2}\underbrace{(r_2^2 - r_1^2)}_{\text{Factor}}\Delta\theta$$

$$= \underbrace{\frac{1}{2}(r_2 + r_1)}_{\bar{r}}\underbrace{(r_2 - r_1)}_{\Delta r}\Delta\theta$$

$$= \bar{r}\Delta r\Delta\theta$$

Ⓒ Area

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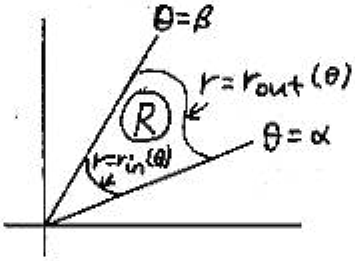


Assume

r_{out}, r_{in} are cont., ≥ 0 on $[\alpha,\beta]$

where $\underbrace{0 \leq \beta-\alpha}_{\text{so } \alpha\leq\beta} \leq 2\pi$

so no "overlapping"

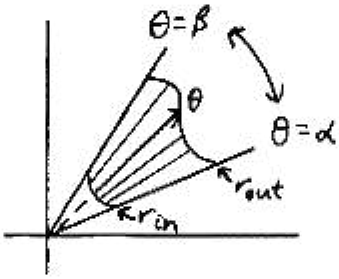
$$\text{Area of } R = \iint_R dA$$

$$= \int_{\theta=\alpha}^{\theta=\beta} \int_{r=r_{in}(\theta)}^{r=r_{out}(\theta)} r\,dr\,d\theta$$

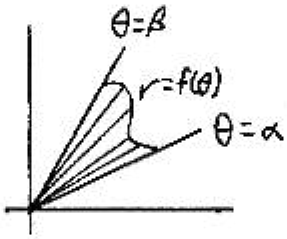
Idea: Aim / Fix θ (outer variable)

Shoot from $r=r_{in}(\theta)$ to $r=r_{out}(\theta)$

Slide / Rotate from $\theta=\alpha$ to $\theta=\beta$



Special Case



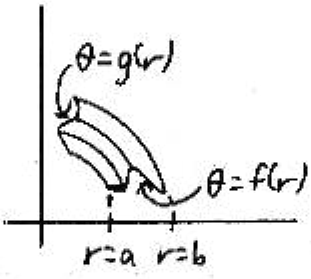
$$\text{Area} = \int_{\theta=\alpha}^{\theta=\beta} \int_{r=0}^{r=f(\theta)} r\,dr\,d\theta$$

$$= \int_{\theta=\alpha}^{\theta=\beta} \left[\tfrac{1}{2} r^2\right]_{r=0}^{r=f(\theta)} d\theta$$

$$= \int_{\theta=\alpha}^{\theta=\beta} \tfrac{1}{2} [f(\theta)]^2\, d\theta$$

Calc II (13.4)

Also



$$\text{Area} = \int_{r=a}^{r=b} \int_{\theta=f(r)}^{\theta=g(r)} r\,d\theta\,dr$$

(D) Volumes, Other ∬s

<u>Ex</u> Find the volume of the solid

bounded above by the graph of $\underbrace{f(x,y)}_{z} = \frac{y^2}{x^2+y^2}$,

bounded below by the xy-plane, and

lying above R,

where R is the region in Quadrant I [of the xy-plane] bounded by the graphs of $x=0$, $y=0$, $y=\sqrt{9-x^2}$, and $y=\sqrt{4-x^2}$.

<u>Sol'n</u>

(1) Express $f(x,y)$ in PCs

<u>Recall</u>

$$x = r\cos\theta$$
$$y = r\sin\theta$$

$$x^2+y^2=r^2$$

$$\underbrace{\frac{y^2}{x^2+y^2}} = \frac{(r\sin\theta)^2}{r^2} = \frac{\cancel{r^2}\sin^2\theta}{\cancel{r^2}} = \underbrace{\sin^2\theta} \quad (\text{if } r\neq 0)$$

Observe: These ≥ 0 for all $(x,y)\neq(0,0)$ (i.e., $r\neq 0$), so the graph never falls below the xy-plane.

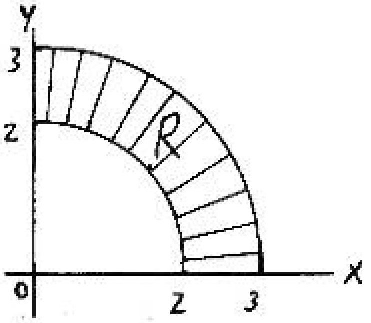
② Graph R (if possible)

$$y = \sqrt{9-x^2} \qquad\qquad y = \sqrt{4-x^2}$$

$$y^2 = 9 - x^2, \quad y \geq 0 \qquad\qquad y^2 = 4 - x^2, \quad y \geq 0$$

$$x^2 + y^2 = 9, \quad y \geq 0 \qquad\qquad x^2 + y^2 = 4, \quad y \geq 0$$

Improper ∬ $r \to 0$



in Quadrant I

part of annulus (ring) ◎

Note These are hard:

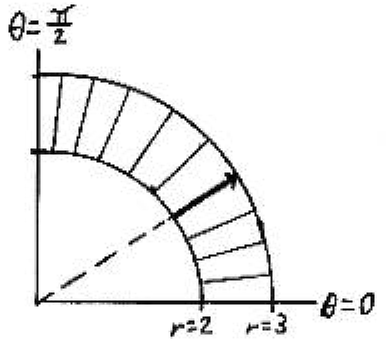
$$\int_{x=0}^{x=2}\int_{y=\sqrt{4-x^2}}^{y=\sqrt{9-x^2}} \frac{y^2}{x^2+y^2}\,dy\,dx + \int_{x=2}^{x=3}\int_{y=0}^{y=\sqrt{9-x^2}} \ldots,$$

Involves $\tan^{-1}$

$$\int_{y=0}^{y=2}\int_{x=\sqrt{4-y^2}}^{x=\sqrt{9-y^2}} \frac{y^2}{x^2+y^2}\,dx\,dy + \int_{y=2}^{y=3}\int_{x=0}^{x=\sqrt{9-y^2}} \ldots,$$

③ Use ② to Set Up the $\iint$ in PCs.

Method I



Aim / Fix θ (outer variable).

Shoot From $r=2$ to $r=3$.

Slide / Rotate From $\theta=0$ to $\theta=\frac{\pi}{2}$.

$$\iint_R \sin^2\theta \, dA$$

$$= \int_{\theta=0}^{\theta=\frac{\pi}{2}} \int_{r=2}^{r=3} \underbrace{\sin^2\theta \cdot r}_{\text{Can separate } \theta, r \text{ into different factors}} \, dr \, d\theta$$

Indep. of θ; Can separate θ, r into different factors

$\Rightarrow$ We can separate the $\int$s !!

$$= \left[\int_{\theta=0}^{\theta=\frac{\pi}{2}} \sin^2\theta \, d\theta\right] \underbrace{\left[\int_{r=2}^{r=3} r \, dr\right]}$$

④ Evaluate

$$= \left[\frac{r^2}{2}\right]_2^3$$

$$= \frac{(3)^2}{2} - \frac{(2)^2}{2}$$

$$= \frac{5}{2}$$

"sin is bad"

$$= \frac{5}{2} \int_0^{\frac{\pi}{2}} \underbrace{\frac{1 - \cos(2\theta)}{2}}_{\text{from a Power-Reducing Identity (PRI)}} \, d\theta$$

from a Power-Reducing Identity (PRI)

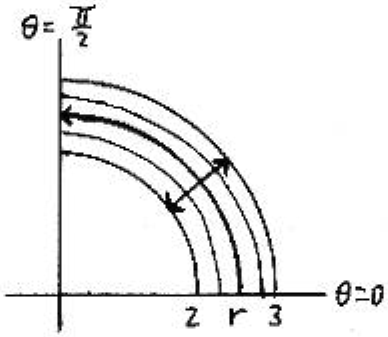
$$= \frac{5}{4}\int_0^{\frac{\pi}{2}} [1 - \underbrace{\cos(2\theta)}_{\text{Use Guess-and-}\checkmark \text{ (or u-sub)}}]\, d\theta$$

$$= \frac{5}{4}\left[\theta - \frac{1}{2}\sin(2\theta)\right]_0^{\frac{\pi}{2}}$$

$$= \frac{5}{4}\left(\left[\frac{\pi}{2} - \frac{1}{2}\underbrace{\sin\left(\cancel{2}\cdot\frac{\pi}{\cancel{2}}\right)}_{=0}\right] - [0]\right)$$

$$= \boxed{\frac{5\pi}{8} \text{ cubic units}}$$

Method 2



Aim/Fix r (outer variable).

Shoot from $\theta = 0$ to $\theta = \frac{\pi}{2}$.

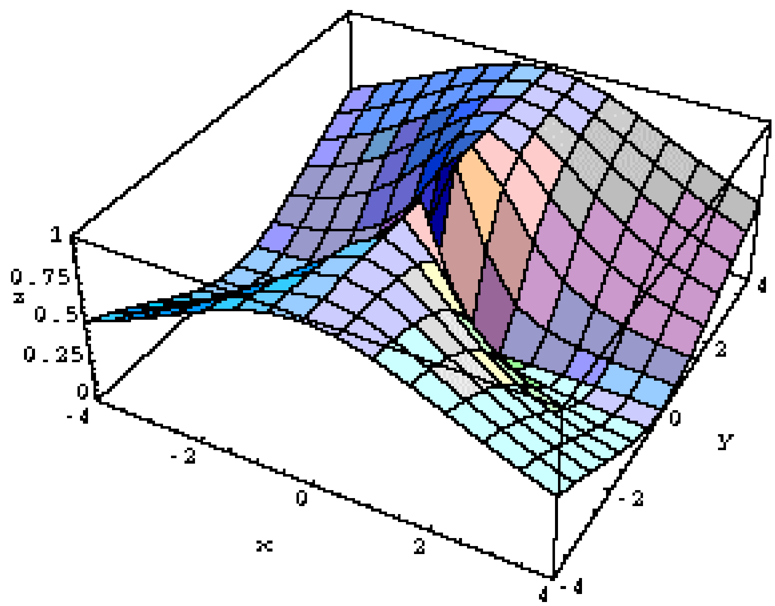
Slide from $r = 2$ to $r = 3$.
Push out

$$\int_{r=2}^{r=3}\int_{\theta=0}^{\theta=\frac{\pi}{2}} \sin^2\theta \cdot r \, d\theta\, dr$$

$= ☆$, also

Here is the graph of $f(x,y)=\dfrac{y^2}{x^2+y^2}$. (Mathematica)

The coordinate axes are rotated a bit differently from what we're used to.



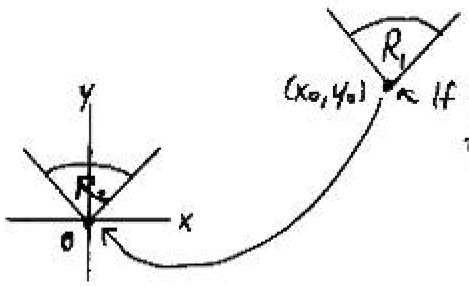
(E) PCs May Help if...

Remember 16.2 on limits

(1) You see "$x^2 + y^2$" in the integrand

(2) R is bounded by circular arcs or rays
(or other polar curves such as cardioids, limaçons, lemniscates, roses, ...)

Note



If this is not the origin/pole, try translating?

We're finding a volume under a surface. You can translate R if you...

We must also translate the surface(s).

$$\iint_{R_1} f(x,y)\,dA = \iint_{R_2} \underbrace{f(x+x_0,\, y+y_0)}\,dA$$

In the $f(x,y)$ rule, replace:
x with $(x+x_0)$ and
y with $(y+y_0)$.

Not "$-$":
O will be our new reference point.

Calc I Idea: $\int_4^6 \underbrace{\sqrt{x-4}}_{f(x)}\,dx = \int_0^2 \underbrace{\sqrt{x}}_{f(x+4)}\,dx$