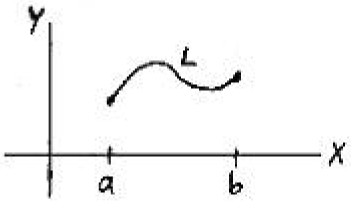
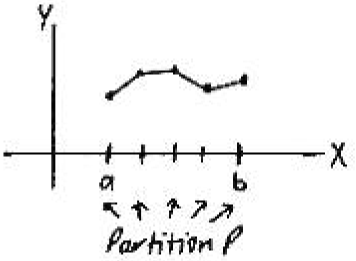
17.4: SURFACE AREA

Ⓐ Idea: Arc Length, "L" (6.5)



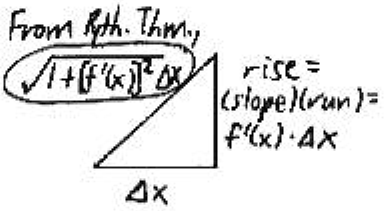


Riemann approx.:

Sample points.
Connect the dots.

We get a piecewise linear
Frankenstein's monster
of pieces of secant lines.

Differentials Idea

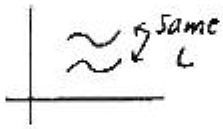


Take the sum of these lengths.

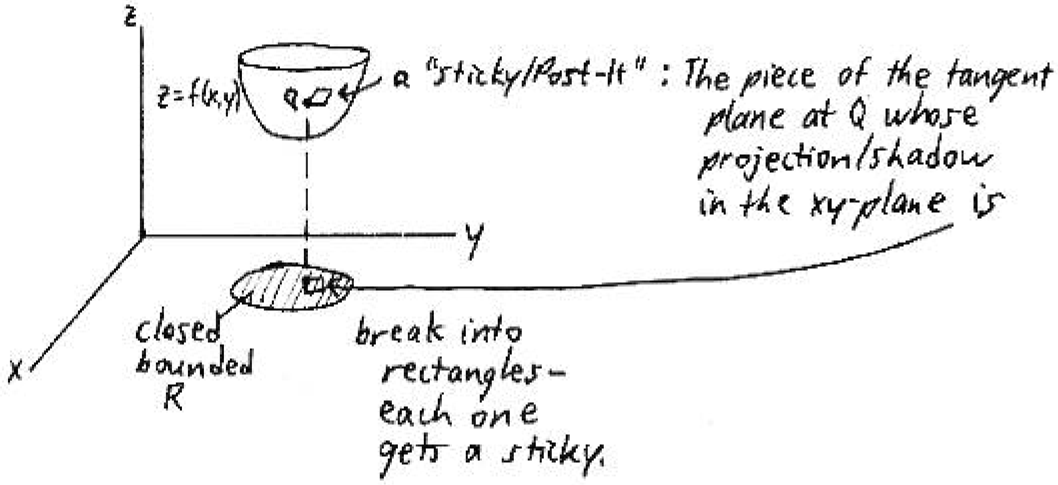
Make partition finer so $\underbrace{\text{widest width}}_{\|P\|,\text{ the norm of partition "}P\text{"}} \longrightarrow 0$.

$$\boxed{L = \int_a^b \sqrt{1+[f'(x)]^2}\,dx}$$

Note: This does not depend directly on $f(x)$:



Ⓑ Idea: Surface Area, "S"

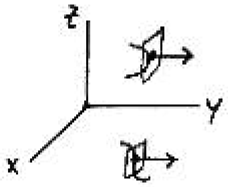


Take the sum of the areas of the stickies.

Make partition finer so $\underbrace{\text{longest diagonal}}_{\|P\|} \to 0$.

Assume f_x, f_y cont. on R
no normal vector to a sticky is $\parallel$ to xy-plane

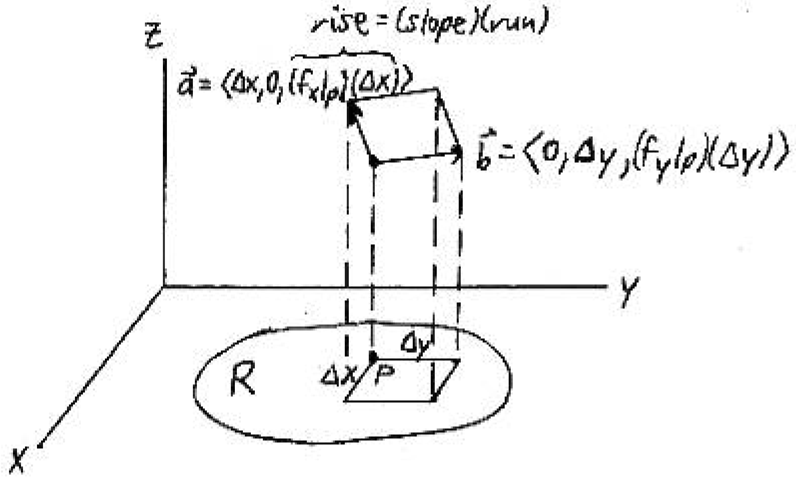
<u>NO</u>



NO $z=f(x,y)$,
could shift perspective

Ⓒ What's the Area of a Sticky?

Again, differentials idea



$$Area = \| \vec{a} \times \vec{b} \|$$

$$= \left\| \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \Delta x & 0 & (f_x|_p)(\Delta x) \\ 0 & \Delta y & (f_y|_p)(\Delta y) \end{vmatrix} \right\|$$

$$= \| \langle -(f_x|_p)\Delta x \Delta y, -(f_y|_p)\Delta x \Delta y, \Delta x \Delta y \rangle \|$$

$$= \| \langle -(f_x|_p), -(f_y|_p), 1 \rangle \| \Delta x \Delta y$$

$$= \sqrt{1 + (f_x|_p)^2 + (f_y|_p)^2} \, \Delta x \Delta y$$

I skip in class.

(D) Formula for S

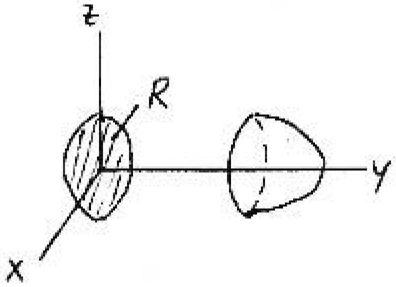
$$\boxed{S = \iint_R \underbrace{\sqrt{1 + [f_x(x,y)]^2 + [f_y(x,y)]^2}\ dA}_{= dS}}$$

Notes ① Looks like extension of arc length to higher dim.

(I don't know if we deal w/ "surface area" in even higher dims.)

② This formula does not depend directly on f, just like for L. same S

③ Can be modified for, say, $y = f(x,z)$.



May need improper $\iint$ See Challenge Problem

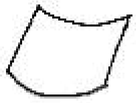
④ If you have a surface of revolution, the Method from 6.5 may be easier. Translations/Rotations may help.

Part of sphere more promising using 6.5 than part of paraboloid, for example.

(I think ellipsoids can be tricky either way.)

Using 17.4, you often use PCs.

(E) Ex

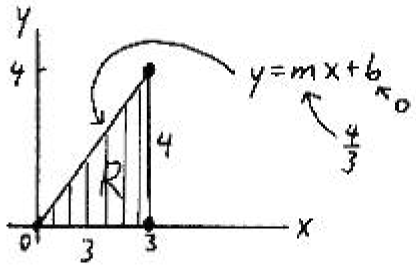


tilted parabolic cylinder

Find the surface area of the graph of $z = x^2 + 4y + 1$ over R, where R is bounded by a triangle with vertices $(0,0)$, $(3,0)$, and $(3,4)$.

Sol'n

R:



$y = \frac{4}{3}x$

$$f(x,y) = x^2 + 4y + 1$$

$$f_x(x,y) = 2x$$
$$f_y(x,y) = 4$$

$$S = \iint_R \sqrt{1 + [f_x(x,y)]^2 + [f_y(x,y)]^2}\ dA$$
$$= \int_{x=0}^{x=3} \int_{y=0}^{y=\frac{4}{3}x} \sqrt{1 + [2x]^2 + [4]^2}\ dy\ dx$$
$$= \int_{x=0}^{x=3} \int_{y=0}^{y=\frac{4}{3}x} \sqrt{4x^2 + 17}\ dy\ dx$$

If we had $\int \sqrt{4x^2+17}\,dx$, use
Table of $\int$s (pp. A21 to A26)
or Trig Sub: $2x = \sqrt{17}\tan\theta$

Tough $\int$? Try: Table of $\int$s (9.7) or Trig Sub (9.3)
Other Ch. 9 Methods: (Parts (9.1), Neat u-sub (9.1), ...)
PCs (is R well-suited?)

$$\int_{x=0}^{x=3} \left[\sqrt{4x^2+17} \cdot y \right]_{y=0}^{y=\frac{4}{3}x} dx$$

$$= \int_{x=0}^{x=3} \left(\sqrt{4x^2+17} \cdot \frac{4}{3}x - \cancel{[0]} \right) dx$$

$$u = 4x^2+17 \qquad \begin{cases} x=0 \Rightarrow u=17 \\ x=3 \Rightarrow u=53 \end{cases}$$

$$du = 8x\,dx$$

$$\Rightarrow \frac{1}{8} du = x\,dx$$

$$= \int_{17}^{53} \sqrt{u} \cdot \frac{\cancel{4}^{1}}{3} \left(\frac{1}{\cancel{8}_{2}} du \right)$$

$$= \frac{1}{6} \int_{17}^{53} u^{1/2}\, du$$

$$= \frac{1}{6} \left[\frac{u^{3/2}}{3/2} \right]_{17}^{53}$$

$$= \frac{1}{\cancel{6}_{3}} \cdot \frac{\cancel{2}^{1}}{3} \left[u^{3/2} \right]_{17}^{53}$$

$$= \frac{1}{9} \left(53^{3/2} - 17^{3/2} \right)$$

$$\approx \boxed{35.08 \text{ sq. units}}$$