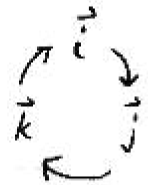
17.5: TRIPLE INTEGRALS ($\iiint$s)

Ⓐ Rotating the Coordinate Axes

Like TNB frame from 15.3

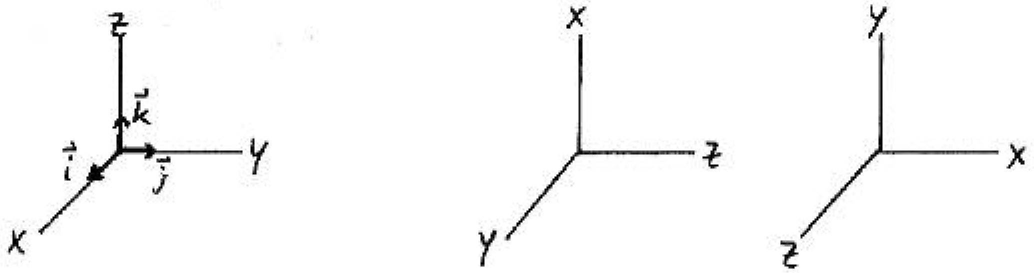
(14.4)



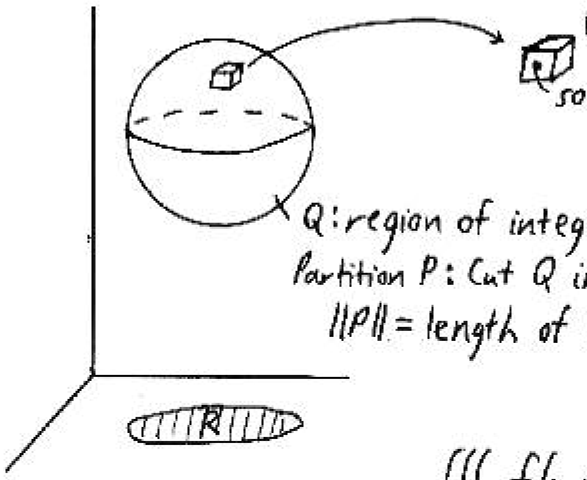
$\vec{i} \times \vec{j} = \vec{k}$ ← thumb from Right-Hand Rule $\qquad \vec{j} \times \vec{k} = \vec{i} \qquad \vec{k} \times \vec{i} = \vec{j}$

Turn the wheel!

+++ (Octant I) opens towards you



Ⓑ Idea



$$\iiint_Q f(x,y,z)\,dV = \lim_{\|P\|\to 0} \sum_{k=1}^{n} \underbrace{f(x_k, y_k, z_k)}_{\text{Assume constant throughout box \#k.}} \Delta V_k$$

Special Case: $\iiint_Q 1\,dV$ = Volume of Q

C Ex

Find $\iiint_Q y \, dV$, where Q is the solid [region] bounded by the graphs of:

You observe

① $y = 0$

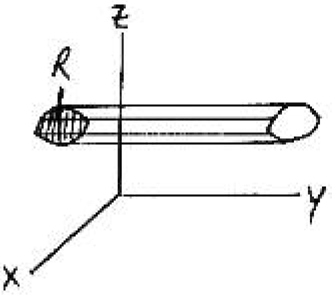
② $y - x^2 = 1$

③ $z = x^2 - 3x$

④ $2x + z = 2$

$\Bigg\}$ Cylinders $\perp$ xz-plane (swept $\parallel$ y-axis)
"We hope they "trap" some space with projection R on the xz-plane.

Ex



Sol'n

Step 1 What is R?

Step 1a Intersection Points

$$\left\{\begin{array}{ll} ③ & z = x^2 - 3x \\ ④ & 2x + z = 2 \Rightarrow z = 2 - 2x \end{array}\right\} \Rightarrow x^2 - 3x = 2 - 2x$$

$$x^2 - x - 2 = 0$$

$$(x-2)(x+1) = 0$$

Optional: Use $z = 2 - 2x$

$$\Rightarrow \begin{array}{l} x = 2 \\ \text{or} \\ x = -1 \end{array} \begin{pmatrix} \Rightarrow z = -2 \\ \Rightarrow z = 4 \end{pmatrix}$$

If do $\overset{z}{+}x$, θ orientation reversed; y points into board

↓counter

↑clockwise

Step 1b In $\overset{x}{+}z$, which graph is on right? left?

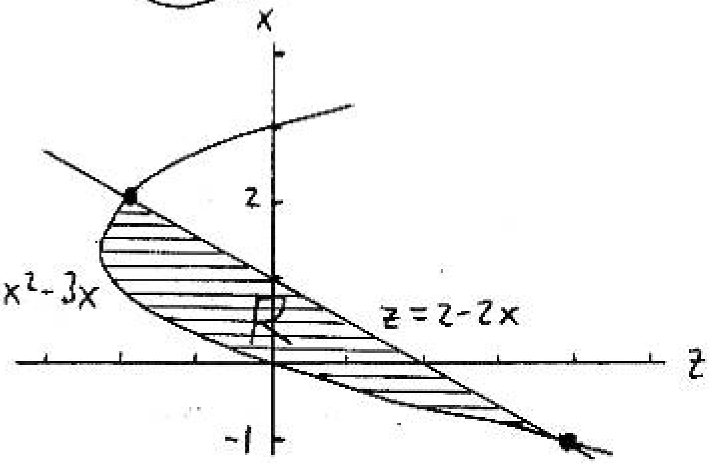
③ $z = x^2 - 3x$ ⇒ parabola opening right ⊂ ← left

④ $2x + z = 2$ ⇒ line ╲ ← right

↑

they intersect, so →

Turns out (Graph R?)



Easier to scan this way (dx: outer)

Step 2 Based on R, Set Up the Outer $\iint$

$$\int_{x=-1}^{x=2} \int_{z=x^2-3x}^{z=2-2x} \left[\int_{y=?}^{y=??} y \, dy \right] dz \, dx$$

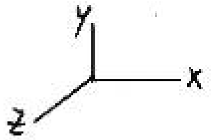
Visualize?

Use symmetry? (Take integrand into account; also inner $\int$s.)

Use PCs?

$x^2 + z^2 = r^2$, $z = r\cos\theta$, $x = r\sin\theta$

Step 3 Innermost $\int$



① $y = 0$ (bottom)

② $y - x^2 = 1 \Rightarrow y = x^2 + 1 \geq 0$ for all x in $[-1, 2]$ (top)

$$\int_{x=-1}^{x=2} \int_{z=x^2-3x}^{z=2-2x} \int_{y=0}^{y=x^2+1} y \, dy \, dz \, dx$$ ← the Set-Up

Step 4 Evaluate

$$= \int_{x=-1}^{x=2} \int_{z=x^2-3x}^{z=2-2x} \left[\frac{y^2}{2} \right]_{y=0}^{y=x^2+1} dz \, dx$$

$$= \int_{x=-1}^{x=2} \int_{z=x^2-3x}^{z=2-2x} \frac{1}{2}\left[(x^2+1)^2 - \cancel{(0)}\right] dz \, dx$$

$$= \frac{1}{2} \int_{x=-1}^{x=2} \left[(x^2+1)^2 z \right]_{z=x^2-3x}^{z=2-2x} dx$$

$$= \frac{1}{2} \int_{-1}^{2} \left(\left[(x^2+1)^2 (2-2x) \right] - \left[(x^2+1)^2 (x^2 - 3x) \right] \right) dx$$

Calc I !!

⋮

$$= \boxed{\frac{225}{28}}$$

D Mass

Old Ex $\displaystyle\iiint_Q \underbrace{y^2}_{} \, dV$ = Mass of Q (i.e, the solid taking up Q)

Let's say this is $\rho(x,y,z)$, or $\delta(x,y,z)$,
a mass density function.
Nonconstant on $Q \Rightarrow$ Solid is nonhomogeneous.

(E) Average Value of f in Q

'temperature!

$$= \frac{\iiint\limits_Q f(x,y,z)\,dV}{\iiint\limits_Q dV}$$ ← Volume of Q

This follows the classic "average" template of

$$\frac{\text{Sum}}{\text{Input Size}}$$ ← "Integration is continuous summation."