17.6: MOMENTS and CENTERS OF MASS

Ⓐ Idea: Weighted Averages (Discrete Case)

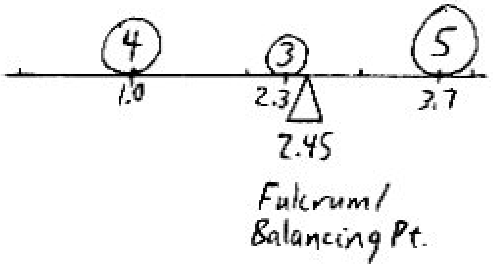
Ex (GPA)

	Grade (x)	# Units (Weights w)
Math	A- (3.7)	5
Chem	D (1.0)	4
Bio	C+ (2.3)	3
		12

$$GPA = \frac{\sum x \cdot w}{\sum w}$$

$$= \frac{(3.7)(5) + (1.0)(4) + (2.3)(3)}{12}$$

$$= 2.45$$



Ⓑ 2D Lamina "L" with Shape R: Center of Mass

(Really, there should be some [constant] thickness.)

Mass of L = "m"

$$= \iint_R \underbrace{\delta(x,y)}_{\text{or } \rho(x,y)\text{: area mass density at } (x,y)} dA$$

(mass per unit area)

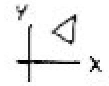
Center of Mass = $(\bar{x}, \bar{y})$, where

$$\bar{x} = \frac{\iint_R x\,\delta(x,y)\,dA}{\iint_R \delta(x,y)\,dA}\begin{array}{l}\Big\}\,M_y\text{, the first moment about the y-axis. Why? (*)}\\[2ex]\Big\}\,m\end{array}$$

This is [literally] a weighted average of the x-coords. throughout R.

Think: x = grade
$\delta(x,y)$ = # units

__Exs__



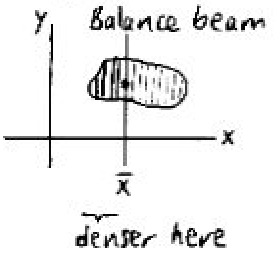
shape favors higher $\bar{x}$, but what about density?

density favors lower $\bar{x}$

(*) M_y measures the tendency to rotate about the y-axis.

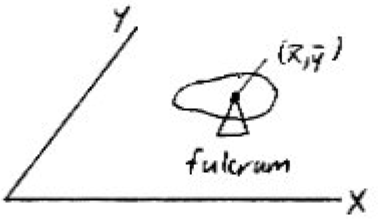
Larson 6.6

1-D __Ex__:
If you want
to rotate a
see-saw,
would you
sit close
to the
fulcrum
or far away?



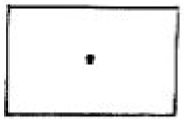
(Imagine mass of L being concentrated at the center of mass.)

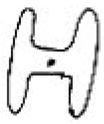
$$\bar{y} = \frac{\left.\iint_R y\,\delta(x,y)\,dA\right\}\ M_x,\ \underline{\text{the first moment about the x-axis.}}}{\left.\iint_R \delta(x,y)\,dA\right\}\ m}$$



If $\delta(x,y)$ = a constant, then L is <u>homogeneous</u>, and $(\bar{x},\bar{y})$ is the <u>centroid</u>, which only depends on shape. (See Section 6.7.)

<u>Ex</u>



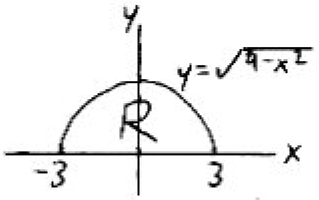


Use symmetry!

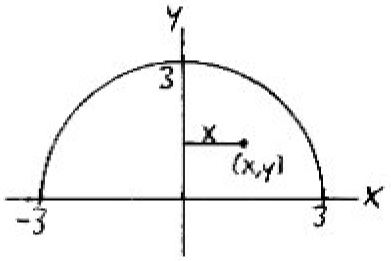
<u>Ex</u> L has shape R, which is bounded by the x-axis and the graph of $y=\sqrt{9-x^2}$. The density at $P(x,y)$ is directly proportional to the square of the distance from P to the y-axis. Find the center of mass for L.

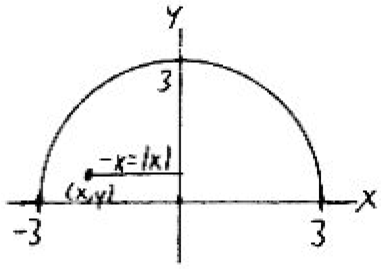
<u>Sol'n</u>

<u>Sketch R</u>



Find Density $\delta(x,y)$





Distance from P to the y-axis = $|x|$
Square of this = x^2

$\boxed{\delta(x,y) = kx^2}$, where $k > 0$

↑constant of proportionality

(don't have to find)

Guess!

Find $\bar{x}$

How to Ace 208

$\delta(x,y) = kx^2$ ← "even in x": $\delta(-x,y) = \delta(x,y)$
$k(-x)^2 = kx^2$
True even if x were missing.

} Symmetry of δ about y-axis ($x=0$).

and

R is symmetric about the y-axis ($x=0$).

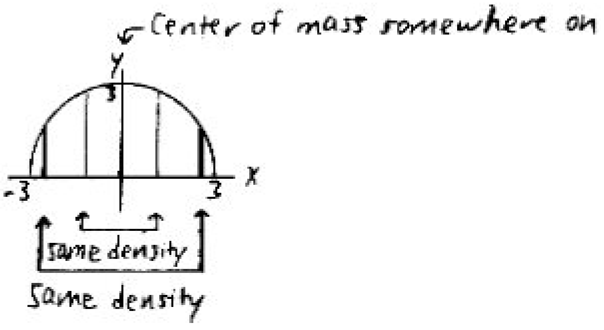
Observe:

$$\bar{x} = \frac{\overbrace{\iint_R x(kx^2)\,dA}^{\text{odd in } x}}{m} = 0$$

Sym.

$\Rightarrow \boxed{\bar{x} = 0}$

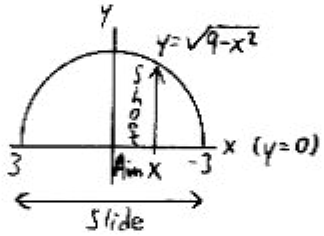
Idea



Find $\bar{y}$

$$\bar{y} = \frac{\iint\limits_R y\,\delta(x,y)\,dA}{\iint\limits_R \delta(x,y)\,dA}$$

R:



$$= \frac{\int_{x=-3}^{x=3}\int_{y=0}^{y=\sqrt{9-x^2}} y\cdot kx^2\,dy\,dx}{\int_{x=-3}^{x=3}\int_{y=0}^{y=\sqrt{9-x^2}} kx^2\,dy\,dx}$$

Set-Up

Don't have to find k!!

$$= \frac{\cancel{k}\int_{x=-3}^{x=3}\left[x^2\cdot\frac{y^2}{2}\right]_{y=0}^{y=\sqrt{9-x^2}}dx}{\cancel{k}\int_{x=-3}^{x=3}\left[x^2 y\right]_{y=0}^{y=\sqrt{9-x^2}}dx}$$

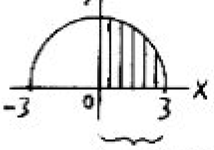
$$= \frac{\int_{x=-3}^{x=3}\frac{1}{2}x^2(\sqrt{9-x^2})^2\,dx}{\int_{x=-3}^{x=3}x^2\sqrt{9-x^2}\,dx}$$

Both integrands are even in x, so $\int_{-3}^{3}\cdots = 2\int_0^3\cdots$

We like to plug in after we find an antiderivative.

$$= \frac{2\int_0^3 \overbrace{\frac{1}{2}x^2(9-x^2)}^{\text{Multiply out}}dx}{\underbrace{2\int_0^3 x^2\sqrt{9-x^2}\,dx}}$$

Use Trig Sub
or Table of $\int$s (p. A22, #31) ← Do



has same $\bar{y}$ as

$$\vdots$$

$$= \frac{\frac{81}{5}}{\frac{81\pi}{16}}$$

$$= \frac{16}{5\pi}$$

Center of mass $= (\bar{x}, \bar{y}) = \boxed{\left(0, \frac{16}{5\pi}\right)}$

Note: $\frac{16}{5\pi} \approx 1.02$. Why so low?

© 2D "L": Moments of Inertia, or Second Moments:

I_x and I_y and I_0

wrt x — wrt y — wrt origin

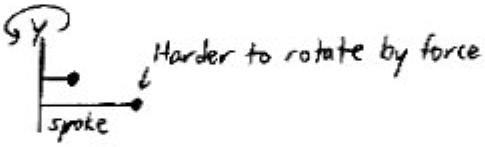
$\underbrace{\text{wrt}}_{\text{with respect to}}$

$$I_y = \iint_R x^2 \delta(x,y)\,dA$$

This is a "weighted sum" of squared distances of points from the y-axis.

Larson: Mass is a measure of resistance to straight-line motion (Calc I)

Higher $I_y \Rightarrow$ Harder to rotate/revolve by external force; it measures resistance to angular momentum.



Cartoon Guide to physics: Moment of inertia is resistance to angular momentum.

$$I_x = \iint_R y^2 \delta(x,y)\,dA$$

$$I_0 = I_x + I_y$$
$$= \iint_R \underbrace{(x^2 + y^2)}_{\substack{= r^2 \\ = \text{sq. dist. from } 0}} \delta(x,y)\,dA$$

Along circle centered at 0?

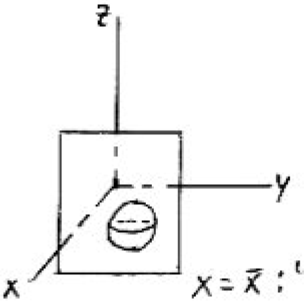
This is the <u>polar moment of inertia</u>.

Ⓓ 3D Solid with Shape "Q": Center of Mass

$(\bar{x}, \bar{y}, \bar{z})$ where

$$\bar{x} = \frac{\left.\iiint_Q x\,\overbrace{\delta(x,y,z)}^{\text{mass density}}\,dV\right\}\ M_{yz}\text{, }\underline{\text{the first moment about the } yz\text{-plane}}\ (*)}{\left.\iiint_Q \delta(x,y,z)\,dV\right\}\ m\text{, the mass of the solid}}$$

(*) M_{yz} measures the tendency to rotate about the yz-plane.
(Is this visualizable?)



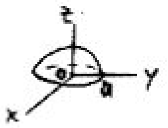
$x = \bar{x}$: "Plane stand" for solid

$\bar{y}, \bar{z}$ analogous

<u>Ex</u> Q: upper hemisphere of radius a w/base centered at O.

$$\delta(x,y,z) = \underbrace{y^4 + 3y^2 + z + 1}_{\text{even in } x\quad \underline{\text{and}}\quad Q \text{ is sym. about } yz\text{-plane}}$$

$\Rightarrow \bar{x} = 0$

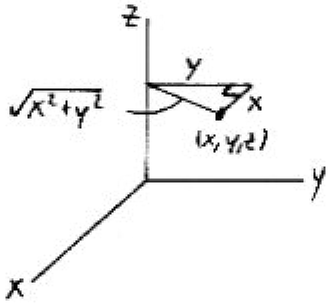


Similarly, $\bar{y} = 0$.

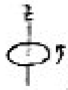
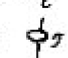
Ⓔ 3D: Moments of Inertia: I_x, I_y, I_z

$$I_z = \iiint_Q (x^2+y^2)\,\delta(x,y,z)\,dV$$ ← Moment of inertia about the z-axis

This is a "weighted sum" of squared distances of points from the z-axis.



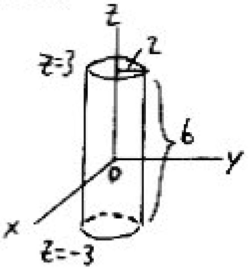
Higher I_z ⇒ Harder to rotate/revolve by external force

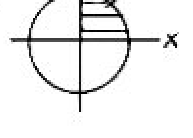
(Think: big wheel  as opposed to )

I_x, I_y analogous

Ex Set up a triple integral for I_y for a cylinder of base radius 2 and height 6 centered at the origin. $\delta(x,y,z) = x^2z^2$.

Sol'n

Q: 

R: 

Can use symmetry of R about x, y-axes, because $\delta(x,y,z) = x^2z^2$ even in y, x.

$$I_y = 4\int_0^2\int_0^{\sqrt{4-x^2}}\underbrace{\int_{-3}^{3}}_{\text{Can use sym., } \delta \text{ even in } z;\ 2\int_0^3}(x^2+z^2)\overbrace{x^2z^2}^{\delta(x,y,z)}\overbrace{dz\,dy\,dx}^{dV}$$

Turns out: $\frac{2664\pi}{5}$

$I_x = \frac{2184\pi}{5}$
$I_z = 142\pi$