17.7: CYLINDRICAL COORDS. (Cyl. Cs)

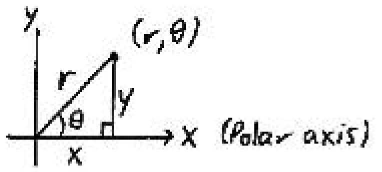
Ⓐ Intro

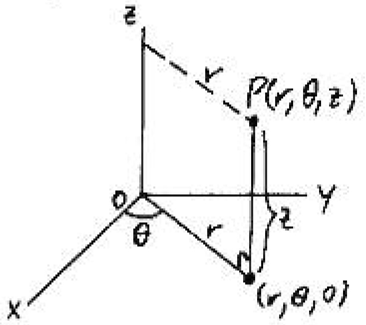
How do we extend PCs to 3D?

Throw in z (or x or y, as the case may be)

if you use PCs to coordinatize the yz-plane

PCs (2D)	Cyl. Cs (3D)





$x = r\cos\theta$

$y = r\sin\theta$

$x^2 + y^2 = r^2$

$\tan\theta = \frac{y}{x}$

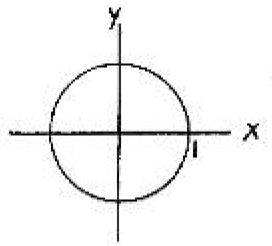
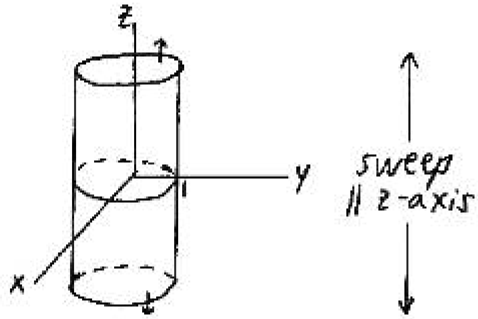
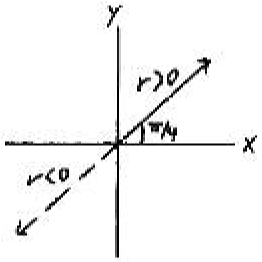
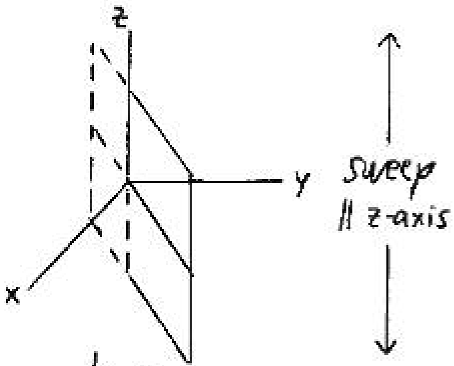
(if $x \neq 0$; Watch Quadrant!)

r = distance of P from z-axis

Ⓑ The Basic Principle of Graphing

> The graph of an equation consists of all points whose coords. satisfy the equation.

Graph:

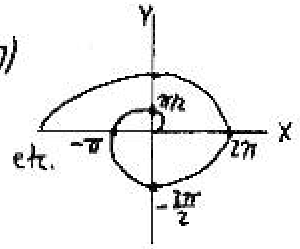
	PCs (2D)	Cyl. Cs (3D)
$r = 1$	 circle	 right circular cylinder {points 1 unit away from the z-axis}
$\theta = \frac{\pi}{4}$	 line	 plane

If θ unrestricted over reals $\Rightarrow$ sym. about y-axis

p.661 – Swok

PCs (2D)

$r = \theta$
$(\theta \geq 0)$



a spiral of Archimedes
$r = a\theta,\ a \neq 0$

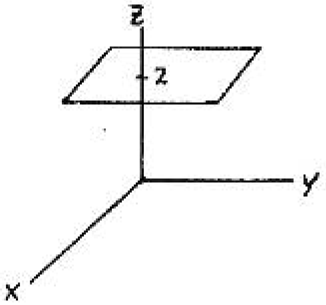
Cyl. Cs (3D)

Sweep $\parallel$ z-axis

Think: Hostess treat Cinnabon

Graph in (3D):

$z = 2$ (same as for Cartesian Cs)



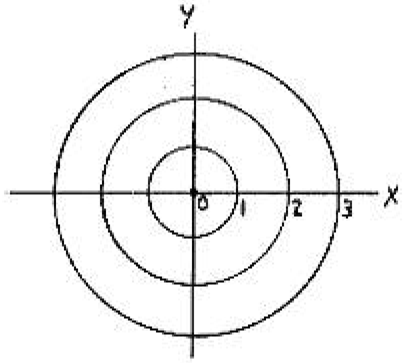
plane

$z = r$ $(r \geq 0)$

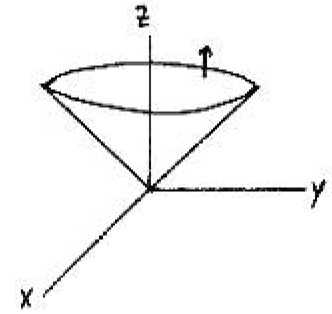
A level curve analysis may help:

$\underbrace{f(r,\theta)}_{=z} = r$

Replace with k



Graph:



$z = r$

$\Rightarrow$ Cartesian Cs

$z^2 = r^2 \quad (z \geq 0)$

$z^2 = x^2 + y^2 \quad (z \geq 0)$

Upper nappe of a cone

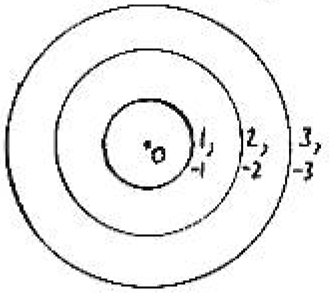
Graph $z = r$ (r unrestricted over reals)

A level curve analysis

"multiple identities"

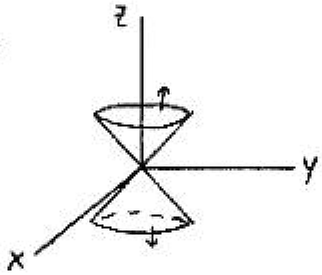
$\underbrace{f(r, \theta)}_{=z} = r$ ← The Vertical Line Test does not apply. (x,y) has multiple PC representations.

LCs can intersect in Cyl. Cs settings.



Lift and drop

Graph:

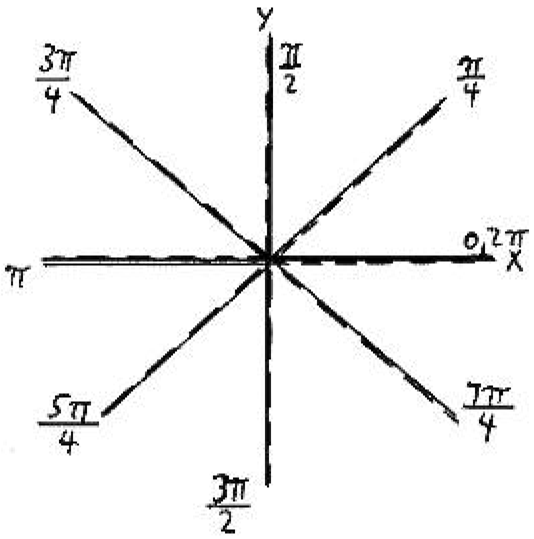


Complete double-napped cone

Graph $z=\theta \quad (0 \le \theta \le 2\pi)$

A level curve analysis

$$\underbrace{f(r,\theta)}_{=z}=\theta$$



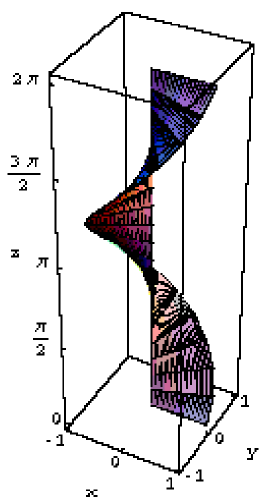
Graph $z = \theta \quad (0 \le \theta \le 2\pi)$.

Think of a rising, rotating line with a rising z-intercept that never intersects the xy-plane (except for $z = 0$). These lines are “parallel” to the xy-plane.

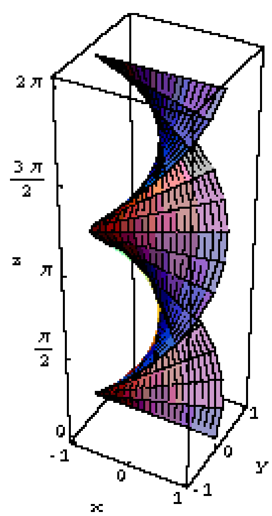
The pole may be coordinatized as $(0, \theta)$ in polar coordinates, where θ is any real number. Here, we have the restriction $0 \le \theta \le 2\pi$. The multiple representations (“identities”) of the pole lead to infinitely many image points along the z-axis of the form $(0, \theta = z, z = \theta)$, or $(0, z, z)$, where $0 \le \theta \,(\text{or } z) \le 2\pi$.

Mathematica graphs:

$0 \le r \le 1$



$-1 \le r \le 1$



© Ex

Describe the graph of $r\cos\theta = \tan\theta + 4$

<u>Sol'n</u> Let's go to Cartesian Cs.

$$r\cos\theta = \tan\theta + 4$$

$$r\cos\theta = \frac{\sin\theta}{\cos\theta} + 4$$

$$r\cos^2\theta = \sin\theta + 4\cos\theta \qquad (\cos\theta \neq 0)$$

<u>Trick</u>: Multiply both sides by r.

$$r^2\cos^2\theta = r\sin\theta + 4r\cos\theta \quad (\text{if } \underbrace{\cos\theta \neq 0}, \underbrace{r \neq 0})$$

Note: $\cos\theta = 0 \Leftrightarrow \tan\theta$ is undefined, so no corresp. points

if $r = 0$, we get $0 = \tan\theta + 4$. See ④ on 17.7.8.

or

$x = \frac{y}{x} + 4$

$(x \neq 0)$

$x = 0 \Leftrightarrow \tan\theta$ und. (unless $y = 0$, also)

$$x^2 = y + 4x$$

$$\underbrace{x^2 - 4x}_{\text{Complete the Square}} = y$$

$$(x^2 - 4x + 4) - 4 = y$$

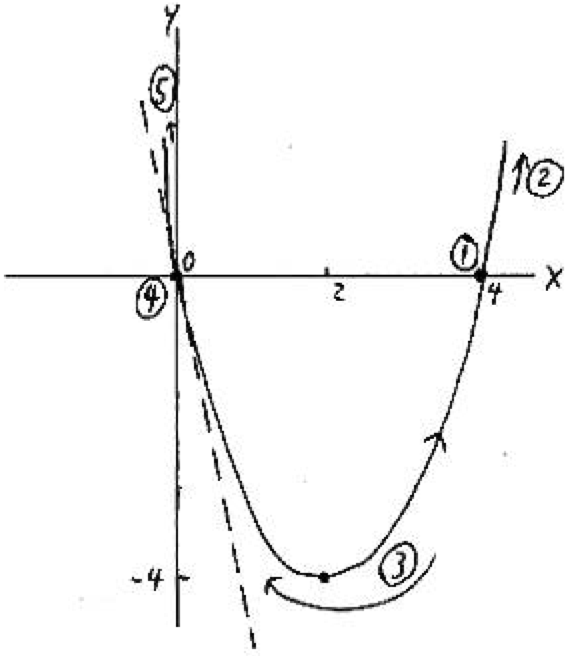
$$y = (x-2)^2 - 4$$

Only z missing $\Rightarrow$ Right cylinder orthogonal to the xy-plane

It's a parabolic cylinder whose xy-trace is an upward-opening ($a = 1 > 0$) parabola with vertex $(2, -4)$.
form: $y = a(x-h)^2 + k$ has vertex (h, k).

(May Skip)

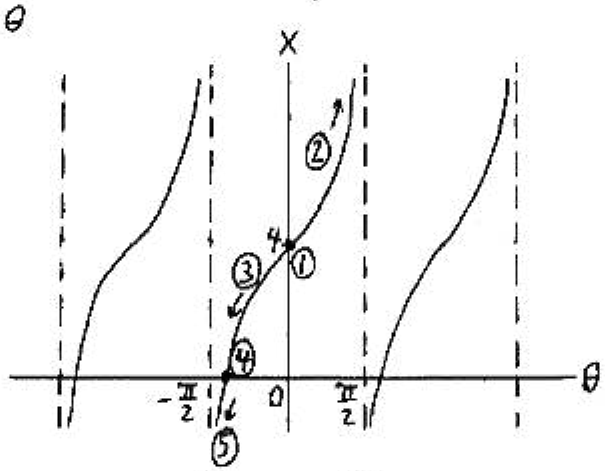
Why does this make sense? Let's focus on the xy-trace.



Note:
$\cos\theta = 0$ at no point, not even the pole.

④ $\theta = \tan^{-1}(-4)$

Graph $\underbrace{x}_{r\cos\theta} = \tan\theta + 4$ using θ and x as Cartesian Cs.



① $\theta = 0 \Rightarrow x = \overbrace{\tan\theta}^{=0} + 4$
$= 4$

② $\theta \nearrow \frac{\pi}{2}^- \Rightarrow x \nearrow \infty$

③ $\theta : 0 \searrow \tan^{-1}(-4)$
$\Rightarrow x : 4 \searrow 0$

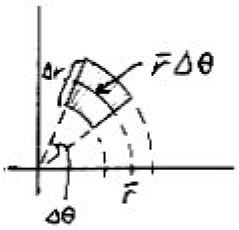
④ $x = 0 \Leftrightarrow \tan\theta = -4$
e.g., $\theta = \tan^{-1}(-4)$

⑤ $\theta \searrow -\frac{\pi}{2}^+ \Rightarrow x \searrow -\infty$

correspond to point at infinity (the parabola is a closed curve passing through this when θ crosses an asymptote; think of wrapping it around a sphere)

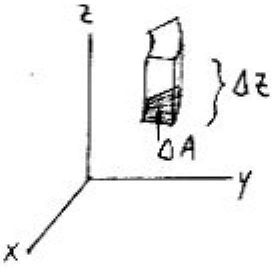
(D) Volume, ΔV, of a Box in Cyl. Cs.

(17.3) Area of a Polar Rectangle



$\Delta A = \bar{r}\Delta r \Delta\theta$

(Now)



$\Delta V = \Delta A \, \Delta z$
$= \bar{r} \Delta r \Delta\theta \Delta z$

$\boxed{dV = r\,dr\,d\theta\,dz}$

or $r\,dz\,dr\,d\theta$ ← more common?
etc.

(E) Ex (#33)

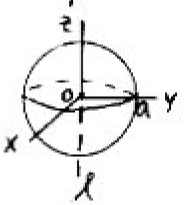
A spherical solid has radius a, and the density at $P(x,y,z)$ is directly proportional to the distance from P to a fixed line ℓ through the center of the solid. Find its mass.

Sol'n

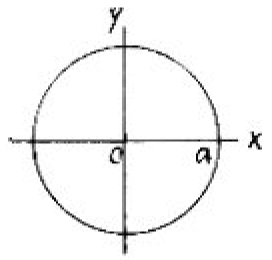
Orient the Sphere "Q"

Let its center be at O.
Let the z-axis be ℓ.

Sketch Q (Optional?)



Sketch R, the projection of Q onto the xy-plane



An easy polar rectangle!

Find density, δ

$\delta = kr \quad (r \geq 0)$

More precisely, $\delta(r,\theta,z)$, but remember that a point has multiple representations in Cyl. Cs.

Find mass, m, of the solid

$$m = \iiint_Q \underbrace{kr}_{\substack{\delta \\ \text{func.}}} \underbrace{dV}_{= r\,dr\,d\theta\,dz}$$

Use R to set up the outer ∬.

$$= \underbrace{\int_{\theta=0}^{\theta=2\pi}}_{} \int_{r=0}^{r=a} \left[\int_{z=?}^{z=??} kr\,dz\right] r\,dr\,d\theta$$

or $4\int_{\theta=0}^{\theta=\pi/2}$ by sym., but it won't matter

may depend on r, θ

Find z-limits based on surfaces.

Boundary of Q: $x^2+y^2+z^2=a^2$

$$r^2+z^2=a^2$$
$$z^2=a^2-r^2$$
$$z=\pm\sqrt{a^2-r^2}$$

correspond to upper, lower hemispheres

$$m=\int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=a}\left[\int_{z=-\sqrt{a^2-r^2}}^{z=\sqrt{a^2-r^2}} kr\,dz\right]r\,dr\,d\theta$$

$$=k\int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=a} r^2\underbrace{\left[2\int_{z=0}^{z=\sqrt{a^2-r^2}}dz\right]}_{\text{by sym. of } Q \text{ about } xy\text{-plane } (z=0),\ \delta \text{ even in } z}dr\,d\theta$$

$$=2\left[z\right]_{z=0}^{z=\sqrt{a^2-r^2}}$$

$$=2\sqrt{a^2-r^2}$$

$$=2k\int_{\theta=0}^{\theta=2\pi}\underbrace{\int_{r=0}^{r=a} r^2\sqrt{a^2-r^2}\,dr}_{\text{Use Table of } \int\text{s, p. A22, \#31}}\,d\theta$$

$$\overset{(u=r)}{\Rightarrow}\left[\frac{r}{8}(2r^2-a^2)\sqrt{a^2-r^2}+\frac{a^4}{8}\sin^{-1}\left(\frac{r}{a}\right)\right]_{r=0}^{r=a}$$

$$=\left[\underbrace{\frac{a}{8}(2a^2-a^2)\underbrace{\sqrt{a^2-a^2}}_{=0}}_{=0}+\frac{a^4}{8}\underbrace{\sin^{-1}\left(\frac{a}{a}\right)}_{=\sin^{-1}(1)=\frac{\pi}{2}}\right]-[0]$$

Note: $\sin^{-1}(0)=0$

$$=\frac{\pi a^4}{16}$$

17.7.12

$$= \cancel{2}k\left(\frac{\pi a^4}{\cancel{16}_{8}}\right)\underbrace{\int_0^{2\pi} d\theta}_{=[\theta]_0^{2\pi} = 2\pi}$$

$$= \frac{\pi k a^4}{\cancel{8}_{4}} \cdot \cancel{2}\pi$$

$$= \boxed{\frac{\pi^2 k a^4}{4}} \text{ [mass units]}$$