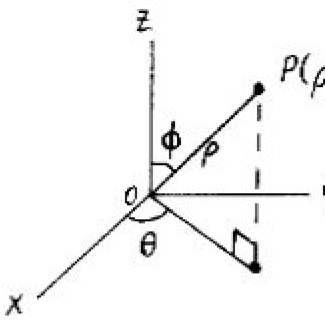
17.8: SPHERICAL COORDS. (SCs)

(A) Intro

How to Ace: same order
Larson: (ρ, θ, ϕ)



Webster's: fī
Everyone else: fee

$\rho \geq 0$, $\phi \geq 0$ hinder exploitation of symmetry

ρ = distance from O to P. $(\rho \geq 0)$
"rho"

ϕ = angle between (nonnegative) positive z-axis, $\overline{OP}$.
"phi" $(0 \leq \phi \leq \pi)$. "Slot machine pullout angle"

θ : same as for Cyl. Cs, but assume $r \geq 0$.

"_" helps w/direction

		# measures of distance	# measures of angles / directions
(x, y, z)	Cartesian Cs	3 } Share z	0
(r, θ, z)	Cyl. Cs	2 }	1 } Share θ
(ρ, ϕ, θ)	SCs	1	2 }

(ρ, ϕ, θ) unique for any pt. off z-axis if θ restricted to $[0, 2\pi)$

ρ, ϕ are unique for P, but θ is not.

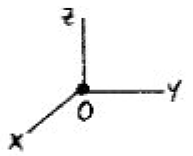
not unique if P is O; there, ϕ can be anything in $[0, \pi]$

source of multiple representations for P
Along z-axis, θ can be any real #.

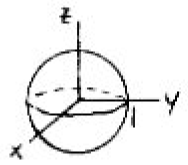
Game: If you fix ρ, ϕ, why does θ locate a point? Etc...
What happens if we "shoot" θ? (SSS issue)

(B) Basic Graphs

$\rho = 0$



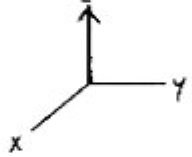
$\rho = 1$



Sphere
(sweep ϕ, θ)

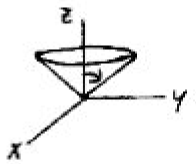
Like a blossoming, then dying flower...

$\phi = 0$



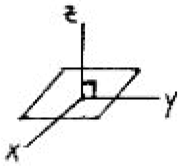
nonnegative
z-axis

$\phi = \frac{\pi}{4}$



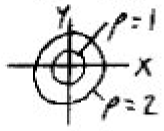
upper nappe
of a cone
"pull out and
spin"

$\phi = \frac{\pi}{2}$

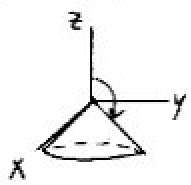


xy-plane
(ρ acts like r)

$\phi = \frac{\pi}{2}$

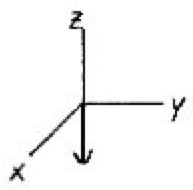


$\phi = \frac{3\pi}{4}$



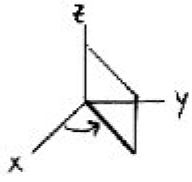
lower nappe
of a cone

$\phi = \pi$



nonpositive
z-axis

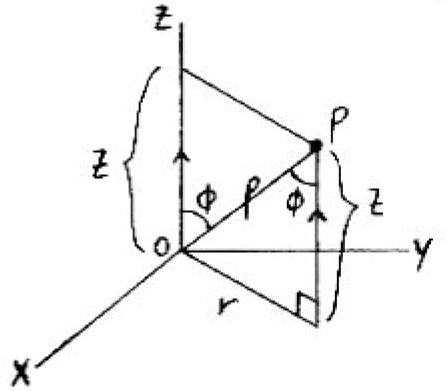
$\theta = \frac{\pi}{4}$



half-plane (full plane for Cyl. Cs)

© Conversions

SCs → Cartesian Cs $(\rho, \phi, \theta) \to (x, y, z)$



If $\rho = 0$, ϕ can be anything in $[0, \pi]$.

$\sin\phi = \frac{r}{\rho} \Rightarrow r = \rho \sin\phi$

PCs: $x = r\cos\theta \Rightarrow$ $\boxed{x = \rho \sin\phi \cos\theta}$

PCs: $y = r\sin\theta \Rightarrow$ $\boxed{y = \rho \sin\phi \sin\theta}$

Why are you so sinful!

$\cos\phi = \frac{z}{\rho} \Longrightarrow$ $\boxed{z = \rho\cos\phi}$

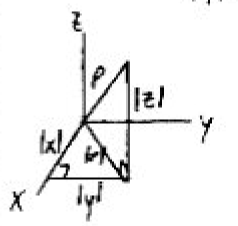
Cartesian Cs → SCs $(x,y,z) \to (\rho, \phi, \theta)$

Find ρ

Idea: Use Distance Formula, Pyth. Thm., Trig IDs (Pyth. IDs).
Try them on SCs for x, y, z.

$\rho^2 = x^2 + y^2 + z^2 \quad (\rho \geq 0)$

$$\boxed{\rho = \sqrt{x^2+y^2+z^2}}$$



$\rho^2 = r^2 + z^2$
$= x^2 + y^2 + z^2$

Find ϕ

$z = \rho \cos \phi$

$\cos \phi = \frac{z}{\rho}$

$$\boxed{\phi = \cos^{-1}\left(\frac{z}{\rho}\right)}$$

Correct, because $\underbrace{0 \leq \phi \leq \pi}_{\text{Range of } \cos^{-1}}$

If $\rho = 0 \Rightarrow P$ is O
$\Rightarrow \phi$ can be anything: $0 \leq \phi \leq \pi$

Find θ

PCs: $\tan \theta = \frac{y}{x} \quad (x \neq 0)$
Watch Quadrant!

there are ∞ many sol'ns. for θ in $\mathbb{R}$

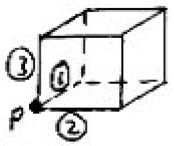
or Use $x = \rho \sin \phi \cos \theta$, or $y = \rho \sin \phi \sin \theta$

Known Found

① Volume of a Spherical Box (Rough analysis!)

Rectangular/Cartesian box:

① Fix y, z; Vary x (here, $x \downarrow$)

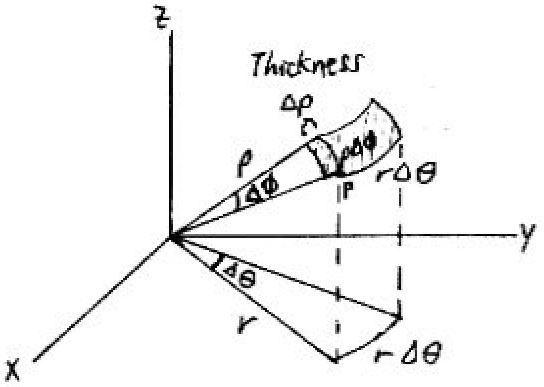


② Fix x, z; Vary y

③ Fix x, y; Vary z

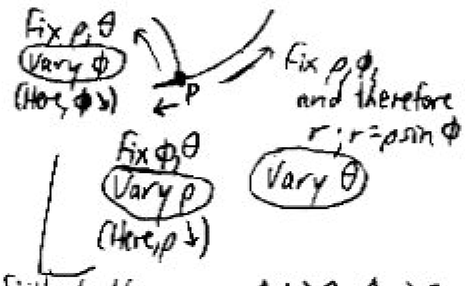
$V = \Delta x \Delta y \Delta z$

ignore that we decreased x; don't worry about $|\Delta x|$



$\Delta\phi$: Think slots arm

$\Delta\theta$: Think roulette wheel



Still, let's say $\Delta\phi > 0$, $\Delta\rho > 0$.

If $\Delta\rho \approx 0$, $\Delta\phi \approx 0$, $\Delta\theta \approx 0 \Rightarrow$ This box $\approx$ Rectangular Box

Volume, $\Delta V \approx (r\,\Delta\theta)(\rho\,\Delta\phi)(\Delta\rho)$

SC: $\rho \sin\phi$

$$\Delta V \approx \rho^2 \sin\phi\, \Delta\rho\, \Delta\phi\, \Delta\theta$$

$$\boxed{dV = \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta}$$

(You'll prove this in 17.9 HW.)

Ⓔ Old Ex (17.7, #33 : See 17.7.9)

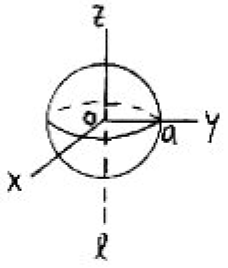
A spherical solid has radius a, and the density at $P(x,y,z)$ is directly proportional to the distance from P to a fixed line ℓ through the center of the solid. Find its mass.

Sol'n

Orient the Sphere "Q"

Let its center be at O.
Let the z-axis be ℓ.

Sketch Q (Optional?)



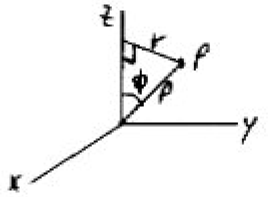
A box in SCs!

Find density, δ

$$\delta = k\sqrt{x^2+y^2} \quad \text{or} \quad kr$$

$x = \rho\sin\phi\cos\theta$ $\quad$ $r = \rho\sin\phi$

$y = \rho\sin\phi\sin\theta$

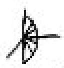


$$= k\rho\sin\phi$$

<u>Find mass, m, of the solid</u>

$$m = \iiint_Q \underbrace{k\rho\sin\phi}_{\delta\text{ func.}} \; \underbrace{dV}_{=\rho^2\sin\phi\, d\rho\, d\phi\, d\theta}$$

Fix θ:



really a half-plane

Then, fix ϕ: Pick a ray

Then, $\rho: 0 \to a$

$$= k\int_{\theta=0}^{\theta=2\pi}\int_{\phi=0}^{\phi=\pi}\int_{\rho=0}^{\rho=a} \underbrace{\rho^3\sin^2\phi}_{\text{separable}}\, d\rho\, d\phi\, d\theta$$

constants, separable $\Rightarrow$ We can separate $\int$s.

$$= k\underbrace{\left[\int_{\theta=0}^{\theta=2\pi} d\theta\right]}_{\substack{=[\theta]_0^{2\pi}\\ =2\pi}}\underbrace{\left[\int_{\phi=0}^{\phi=\pi}\sin^2\phi\, d\phi\right]}_{\text{Use a PRI.}}\underbrace{\left[\int_{\rho=0}^{\rho=a}\rho^3\, d\rho\right]}_{\substack{=\left[\frac{\rho^4}{4}\right]_0^a\\ =\frac{a^4}{4}}}$$

Use a PRI.

$$= \int_0^\pi \frac{1-\cos(2\phi)}{2}\, d\phi$$

$$= \frac{1}{2}\left[\phi - \frac{1}{2}\sin(2\phi)\right]_0^\pi$$

$$= \frac{1}{2}\left(\left[\pi - \frac{1}{2}\underbrace{\sin(2\pi)}_{=0}\right] - [0]\right)$$

$$= \frac{\pi}{2}$$

$$= k(\cancel{2}\pi)\left(\frac{\pi}{\cancel{2}}\right)\left(\frac{a^4}{4}\right)$$

$$= \boxed{\frac{\pi^2 k a^4}{4}}$$

Same as before! Easier!

We didn't have complete separability before. Here, using SCs, all limits of $\int$ are constants.