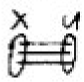
17.9: CHANGE OF VARIABLES and JACOBIANS

Ⓐ Calc I: A New Look at u-Subs

Ex $\int_1^2 e^{3x}\,dx$

Let $\boxed{u=3x} \xrightarrow[\text{Inverse Idea}]{\text{Solve for } x} \boxed{x=\frac{1}{3}u}$ ← We can think of the sub. this way: $x=f(u)$

(u(x): 1-1 func.



)

$$du = \underset{\underset{\boxed{\frac{du}{dx}}}{\uparrow}}{3}\,dx \qquad dx = \underset{\underset{\boxed{\frac{dx}{du}}}{\uparrow}}{\frac{1}{3}}\,du$$

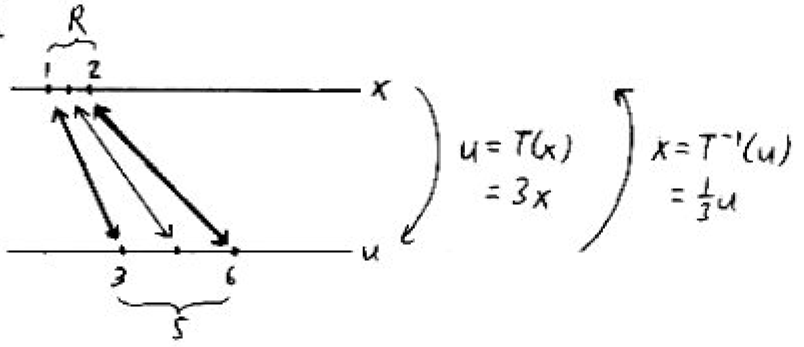
$$\left.\frac{dx}{du}\right|_u = \frac{1}{\left.\frac{du}{dx}\right|_x}$$

corresponding values

Change the limits of ∫

$$x=1 \Rightarrow u(1)=3$$
$$x=2 \Rightarrow u(2)=6$$

Idea



T is a 1-1 transformation of coordinates.

1-1 correspondence between x-values in R, u-values in S.

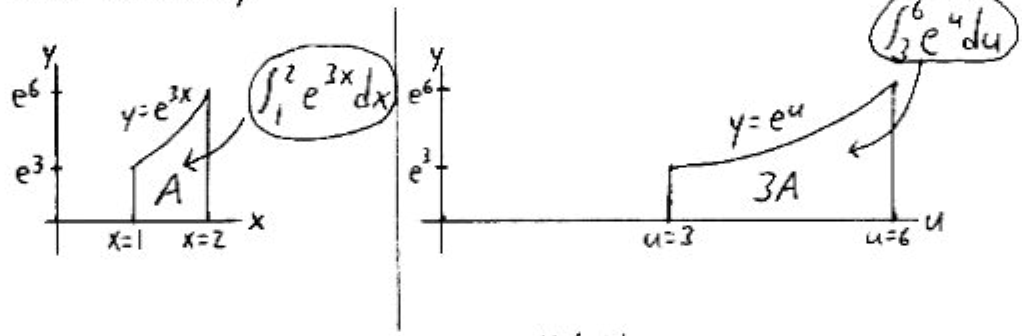
17.9.2

$$\int_1^2 e^{3x}\,dx = \int_{\underset{u(1)}{3}}^{\overset{u(2)}{6}} e^u \cdot \boxed{\tfrac{1}{3}}\,du$$

Why do we need $\boxed{\frac{dx}{du}}$?

Without it...

(Not to scale)



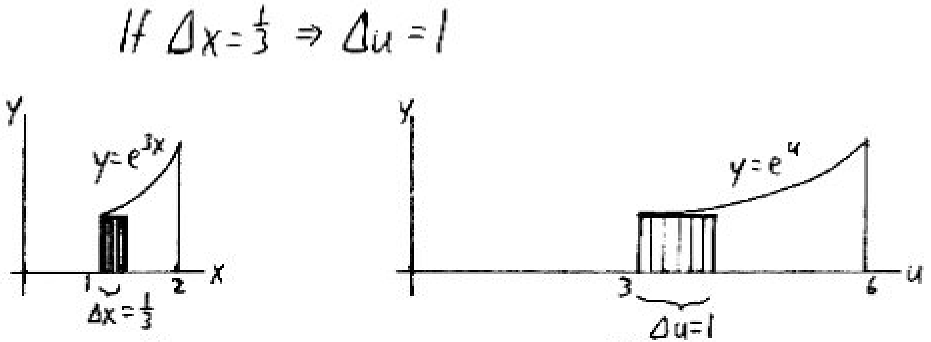
We'd get 3 times
the area!! ☆
We need to
<u>compensate</u>
with a $\frac{1}{3}$.

<u>Why ☆?</u>

Think: Riemann rectangles

$du = 3dx$
$\Delta u = 3\Delta x$

If $\Delta x = \frac{1}{3} \Rightarrow \Delta u = 1$



Rectangles stretch along w/ horizontal axis →

$\longrightarrow \quad \frac{du}{dx} = 3$

$= $ stretching factor $x \to u$ (Think: $\frac{\boxed{du}}{\boxed{dx}}$)

$\longleftarrow \quad \frac{dx}{du} = \frac{1}{3}$

$=$ compensation factor

$=$ stretching factor $u \to x$ (compression, since $0 < \frac{1}{3} < 1$)

$$\int_1^2 e^{3x}\,dx = \int_3^6 e^u \cdot \boxed{\tfrac{1}{3}}\,du$$

Need $\boxed{\frac{dx}{du}}$

If you use $\left|\frac{dx}{du}\right|$, you can adopt the convention $\int_{\text{lower \#}}^{\text{upper \#}}$

Ex $\int_1^2 e^{-3x}\,dx = \int_{-3}^{-6} e^u \cdot \overbrace{\left(-\tfrac{1}{3}\right)}^{dx/du} du$

$= \int_{-6}^{-3} e^u \cdot \left|-\tfrac{1}{3}\right| du$

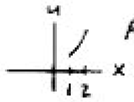
Idea: $\int_b^a \sim = -\int_a^b \sim$

Ex $\int_1^2 x e^{x^2}\,dx$

Let $\boxed{u = x^2} \Longrightarrow \boxed{x = \underset{\cancel{-}}{+}\sqrt{u}}$

x in [1,2]

(u(x): 1-1 func. on [1,2]



passes HLT)

$dx = \underbrace{\boxed{\frac{1}{2\sqrt{u}}}}_{\boxed{\frac{dx}{du}}} du$

$du = \boxed{2x}\,dx$

$\boxed{\frac{du}{dx}}$ = <u>instantaneous</u> stretching factor. ⍟ It changes as x changes.

Note:
$\frac{du}{dx} = 2x$
$\Rightarrow \frac{dx}{du} = \frac{1}{2x}$
$= \frac{1}{2\sqrt{u}}$

$$\int_1^2 x e^{x^2}\,dx = \int_1^4 \sqrt{u}\, e^u \cdot \boxed{\frac{1}{2\sqrt{u}}}\,du$$

u(2)

u(1)

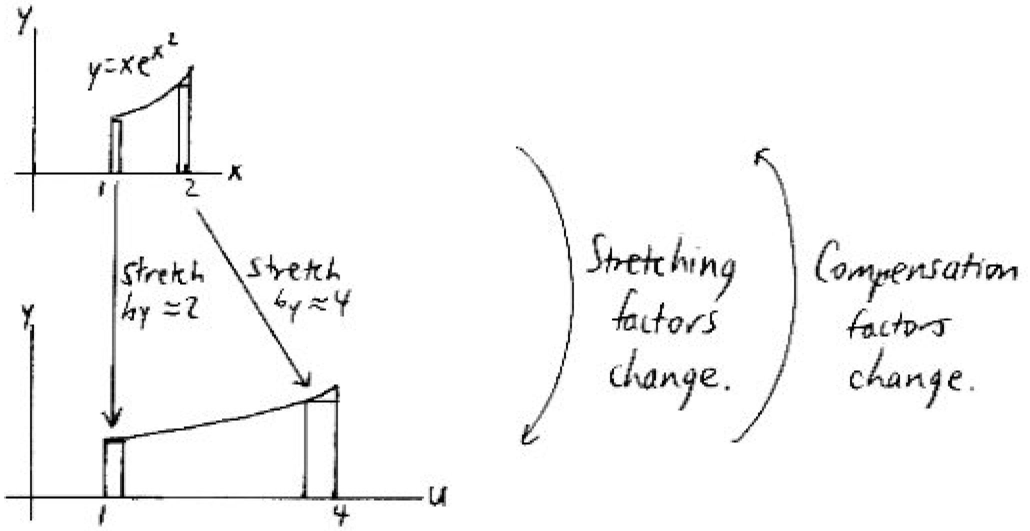
Need $\boxed{\frac{dx}{du}}$ as an <u>instantaneous</u> compensation factor. It changes as u changes.

Ⓐ Idea: Riemann rectangles

$du = 2x\,dx$
$\Delta u \approx 2x\,\Delta x$

If $x \approx 1$, then $\Delta u \approx 2\Delta x$.

If $x \approx 2$, then $\Delta u \approx 4\Delta x$.



Stretching factors change.

Compensation factors change.

Image: We're stretching the x-axis like a piece of taffy in which some parts are stretched more than others.
The corresponding rectangles are stretched in the same way.

Carl Jacobi
(German,
1804-1851)

Ⓑ Jacobians

are compensation factors for multiple $\int$s when we change variables to

① Simplify the region of integration, and/or

② Simplify the integrand.

$\int$ If $x = f(u)$,

Then, $dx = \left|\frac{dx}{du}\right| du$, if you always $\int_{\text{lower \#}}^{\text{higher \#}}$.

$\iint$ If $\begin{cases} x = f(u,v), \\ y = g(u,v) \end{cases}$ have cont. 1st-order PDs where we care

Hard proof:
Larson 6ed
p. 476

$$\text{Then, } dA = \left| \begin{array}{|cc|l} \overset{x}{\downarrow} & \overset{y}{\downarrow} & \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \leftarrow u \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \leftarrow v \end{array} \right| du\, dv$$

abs. value

determinant
= the Jacobian of x and y wrt u and v

$$= \text{“}\frac{\partial(x,y)}{\partial(u,v)}\text{”} \leftarrow x,y \text{ on top}$$

We require that this is never 0 where we care.
what would it compensate for?

so it can't change sign (assume 1st-order PDs cont.)

Stretches/compresses a 2-D region, (corresponding 3-D solid)

<u>Note</u> $\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix}$

OK to switch: $|\text{matrix } A| = |A^T|$

transpose: switch rows, cols

Key: $\frac{\partial}{\partial}$ ← x,y on top ← u,v on bottom

ⓈⓈⓈ If $\begin{cases} x = f(u,v,w) \\ y = g(u,v,w) \\ z = h(u,v,w) \end{cases}$

$$\text{Then, } dV = \left| \underbrace{\begin{array}{ccc|l} \overset{x}{\downarrow} & \overset{y}{\downarrow} & \overset{z}{\downarrow} & \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} & \leftarrow u \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} & \leftarrow v \\ \frac{\partial x}{\partial w} & \frac{\partial y}{\partial w} & \frac{\partial z}{\partial w} & \leftarrow w \end{array}}_{\text{“}\frac{\partial(x,y,z)}{\partial(u,v,w)}\text{”}} \right| du\, dv\, dw$$

How to Ace

If $\vec{r} = \langle x, y, z \rangle$, then this $= \left\| \begin{array}{c} \leftarrow \frac{\partial \vec{r}}{\partial u} \rightarrow \\ \leftarrow \frac{\partial \vec{r}}{\partial v} \rightarrow \\ \leftarrow \frac{\partial \vec{r}}{\partial w} \rightarrow \end{array} \right\|$

$= \left| \text{TSP of } \frac{\partial \vec{r}}{\partial u}, \frac{\partial \vec{r}}{\partial v}, \frac{\partial \vec{r}}{\partial w} \right|$

= Volume of parallelepiped determined by

Stretches/compresses a 3-D region.

Ⓒ A New Look at PCs

Ex A region R in the xy-plane consists of points (x,y):

$$\left.\begin{aligned} x &= \underbrace{r\cos\theta}_{f(r,\theta)} \\ y &= \underbrace{r\sin\theta}_{g(r,\theta)} \end{aligned}\right\}$$

We're expressing the old vars. in terms of the new vars.

This turns out to be easier for us!

where $\left.\begin{aligned} 1 \le r \le 2, \\ \tfrac{\pi}{6} \le \theta \le \tfrac{\pi}{3} \end{aligned}\right\}$ determine a region S in the $r\theta$-plane

What is dA?

$$dA = \left\| \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} \end{vmatrix} \right\| dr\, d\theta$$

$$= \left\| \begin{vmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{vmatrix} \right\| dr\, d\theta$$

$$= \left| r\cos^2\theta + r\sin^2\theta \right| dr\, d\theta$$

$$= \left| r \underbrace{(\cos^2\theta + \sin^2\theta)}_{=1} \right| dr\, d\theta$$

$$= |r|\, dr\, d\theta$$

Given: $1 \le r \le 2$

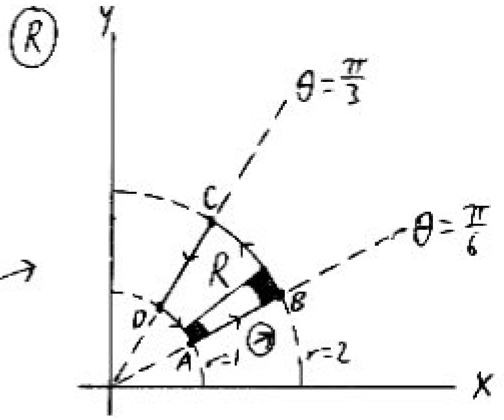
$$= r\, dr\, d\theta$$ ☺

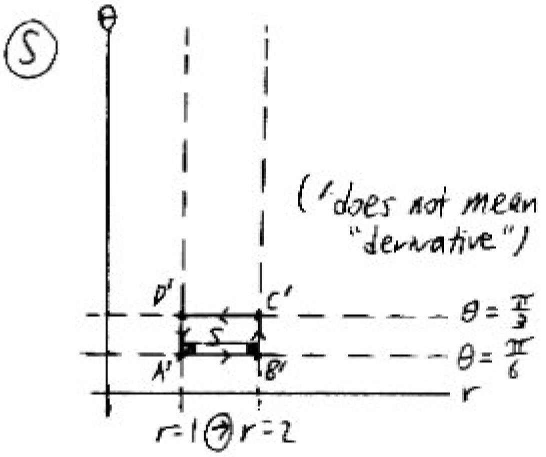
ⓡ $dr\,d\theta$

⇒ Greater stretching of areas as $\boxed{r \to 2}$ (Really compensating for ↝)

Graph R, S

Arrows indicate positive orientation: R always on the left





(' does not mean "derivative")

These are r-curves ($r = \#$), θ-curves ($\theta = \#$) in the xy-plane.

Jacobian turns out to be positive in value, so positive orientation is retained.

Think: Level curves.

Note 1: We like that the boundaries of R, S are piecewise smooth, simple, closed, bounded curves.

has pieces parameterizable by param'ns where derivs. cont., $\neq \vec{0}$ except maybe at endpts.

no ∞   no @   no <···

Note 2: We like that, as we trace the boundary of R once in one direction, we trace the boundary of S once in one direction.

Note 3: Had the Jacobian been negative in value, the orientation would have been reversed along the S-boundary.

Note 4: We like that S is simpler than R. The change of variables may help!

See Larson 6ed, pp. 975-8

① Ex

Let R be the region bounded by:

$$x-2y=0$$
$$x-2y=2$$
$$x+y=1$$
$$x+y=3$$

Evaluate $\iint\limits_R (x+y)\sin(x-2y)\,dA$.

Sol'n

① Change of variables

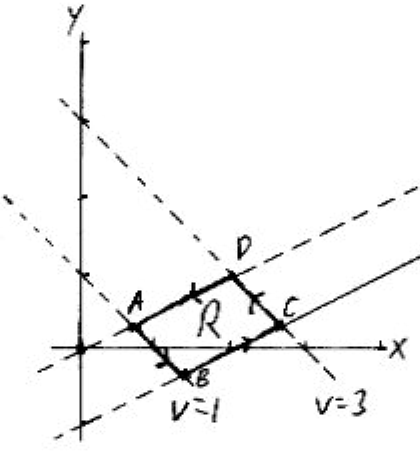
Let $\begin{cases} u = x-2y \\ v = x+y \end{cases}$ or

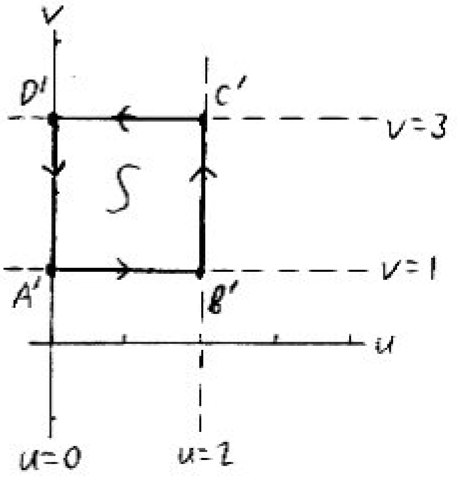
②

② New limits of $\iint$

$u=0$
$u=2$
$v=1$
$v=3$

Note: What do R and S look like?





Jacobian turns out to be positive in value, so positive orientation is retained.

17.9.10

③ Solve for x,y in terms of u,v (Can skip if do ④ next page)

$$\begin{aligned} u &= x - 2y \\ v &= x + y \quad \leftarrow \cdot(-1) \end{aligned}$$

$$\begin{aligned} u &= x - 2y \\ -v &= -x - y \\ \hline u - v &= -3y \end{aligned} \quad \Big\downarrow \text{Add}$$

$$\boxed{y = -\tfrac{1}{3}(u-v) \text{ or } \tfrac{1}{3}v - \tfrac{1}{3}u}$$

Find x

$$\begin{aligned} v &= x + y \\ v &= x - \tfrac{1}{3}(u-v) \\ x &= v + \tfrac{1}{3}(u-v) \end{aligned}$$

$$\boxed{x = \tfrac{1}{3}u + \tfrac{2}{3}v \text{ or } \tfrac{1}{3}(u+2v)}$$

Another Method (because of linearity)

$$\begin{bmatrix} u \\ v \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix}}_{A} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} u \\ v \end{bmatrix}$$

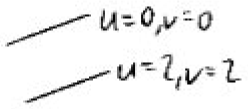
$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{\underset{\text{det}(A)}{\boxed{3}}} \begin{bmatrix} \overset{\text{switched}}{1} & 2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} \quad \leftarrow \text{flipped signs}$$

$$\boxed{\begin{cases} x = \tfrac{1}{3}(u+2v) \\ y = \tfrac{1}{3}(-u+v) \end{cases}}$$

Note If we had done something dumb, like

$$\begin{cases} u = x - 2y \\ v = x - 2y \end{cases},$$

then $\det\left(\begin{bmatrix} 1 & -2 \\ 1 & -2 \end{bmatrix}\right) = 0.$



④ Find Jacobian

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$x = \tfrac{1}{3}u + \tfrac{2}{3}v$$
$$y = -\tfrac{1}{3}u + \tfrac{1}{3}v$$

$$= \begin{vmatrix} \frac{1}{3} & -\frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} \end{vmatrix}$$

$$= \tfrac{1}{9} + \tfrac{2}{9}$$

$$= \boxed{\tfrac{1}{3}}$$

or ④* (Can skip ③)

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$u = x - 2y$$
$$v = x + y$$

$$= \begin{vmatrix} 1 & -2 \\ 1 & 1 \end{vmatrix}$$

$$= 3$$

← If this had x and/or y in it, express in terms of u,v

$$\Rightarrow \frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{\frac{\partial(u,v)}{\partial(x,y)}}$$

$$= \boxed{\tfrac{1}{3}}$$

⑤ Set up ∬, and Evaluate

$$\iint_R \underbrace{(x+y)\sin(x-2y)}_{\text{cont. on } R}\, dA$$

$$= \iint_S v \sin u \cdot \underbrace{\left|\frac{\partial(x,y)}{\partial(u,v)}\right|}_{=\boxed{\frac{1}{3}}}\, \underset{\text{or}}{dv\,du}$$

$$= \frac{1}{3}\int_{u=0}^{u=2}\int_{v=1}^{v=3} v \sin u \; dv\,du$$

$$= \frac{1}{3}\underbrace{\int_0^2 \sin u\, du}_{\substack{=[-\cos u]_0^2 \\ = -\cos 2 + \underbrace{\cos 0}_{=1} \\ = 1-\cos 2}}\ \underbrace{\int_1^3 v\, dv}_{\substack{=\left[\frac{v^2}{2}\right]_1^3 \\ = \frac{9}{2}-\frac{1}{2} \\ = 4}}$$

Note

$$\begin{tikzcd} R \arrow[r, bend left, "\text{Stretching factor: } 3"] & S \arrow[l, bend left, "\text{Compensating factor: } \frac{1}{3}"] \end{tikzcd}$$

$$\text{Area}(S) = 3 \cdot \text{Area}(R)$$

(Remember: $\int_{\text{lower \#}}^{\text{higher \#}}$)

$$= \frac{1}{3}(1-\cos 2)(4)$$

$$= \boxed{\frac{4(1-\cos 2)}{3}}$$

$$\approx 1.8882$$

Ex (#20)

$$\iint_R (3x-4y)\,dx\,dy$$

Boundary of R: $y=3x$, $y=\frac{1}{2}x$, $x=4$

Change of vars.: $x=u-2v$
$y=3u-v$

Sol'n

Rewrite the Integrand:

$$3x-4y = 3(u-2v)-4(3u-v)$$
$$= \boxed{-9u-2v}$$

Jacobian:

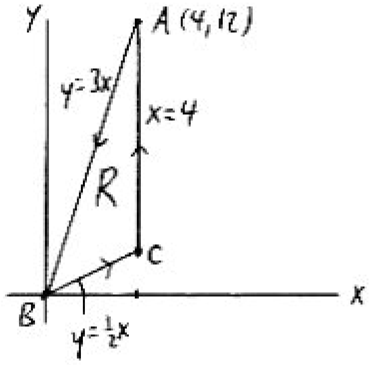
$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & -2 \\ 3 & -1 \end{vmatrix}$$

$$= ⑤$$

17.4.13

What's R?



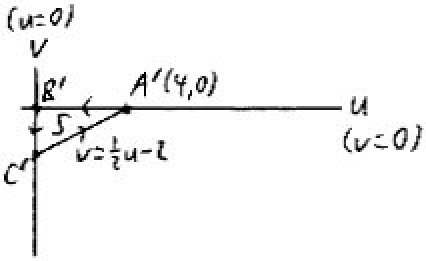
What's S?

Boundaries:

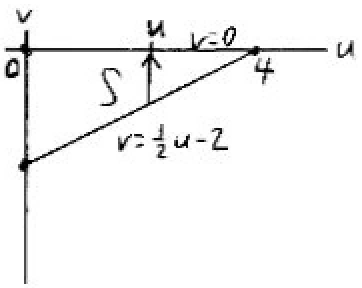
$$y = 3x \qquad y=\tfrac{1}{2}x \qquad x=4$$

$$(3u-v)=3(u-2v) \qquad (3u-v)=\tfrac{1}{2}(u-2v) \qquad (u-2v)=4$$

$$\boxed{v=0} \qquad \boxed{u=0} \qquad \boxed{v=\tfrac{1}{2}u-2}$$



Close-Up:



Note: $A(\overset{x}{4},\overset{y}{12}) \Rightarrow A'(\overset{u}{4},\overset{v}{0})$

← from this graph, or

Solve: $\begin{cases} 4=u-2v \\ 12=3u-v \end{cases}$

$\Rightarrow (u,v)=(4,0)$

Note: Area (R) = 5 · Area (S)

Area (S) = $\frac{1}{5}$ · Area (R)

17.9.14

$$\iint_R (3x-4y)\,dx\,dy = \iint_S (-9u-2v)\cdot \underbrace{\left|\frac{\partial(x,y)}{\partial(u,v)}\right|}_{=|5|} \underbrace{du\,dv}_{\text{or } dv\,du}$$

$$= 5\iint_S (-9u-2v)\,dv\,du$$

Setup →

$$= 5\int_{u=0}^{u=4}\int_{v=\frac{1}{2}u-2}^{v=0} (-9u-2v)\,dv\,du$$

↑Remember, $\int_{\text{lower value}}^{\text{higher value}}$

No more unusual than $\int_{-3}^{0}\sim$.

$$\vdots$$

$$= \boxed{-\frac{640}{3} \quad (= -213.\bar{3})}$$