CH. 18: VECTOR CALCULUS

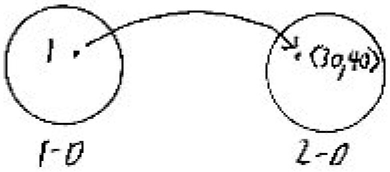
, We'll look at extensions of the fund. Thm. of Calculus (FTC) from Calc I !!

18.1: VECTOR FIELDS

Ⓐ Intro

Ch. 15

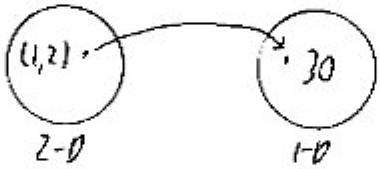
VVF $\vec{r}: D$, a subset of $\mathbb{R} \longrightarrow V_n$ (or $\mathbb{R}^n$)



$\vec{r}(1) = \langle 30, 40 \rangle$

Ch. 16

Scalar func. $f: D$, a subset of $\mathbb{R}^n \longrightarrow \mathbb{R}$

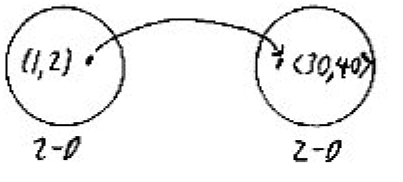


$f(1,2) = 30$

Now

A vector field is described by a

VVF $\vec{F}: D$, a subset of $\mathbb{R}^n \longrightarrow V_n$ (or $\mathbb{R}^n$)



$\vec{F}(1,2) = \langle 30, 40 \rangle$

We use P in $\vec{F} = \langle M, N, P \rangle$ in $\mathbb{R}^3$.

Every point A in a region D gets a vector $\vec{F}|_A$.

(x,y) $\qquad$ $\vec{F}(x,y)$

(x,y,z) $\qquad$ $\vec{F}(x,y,z)$

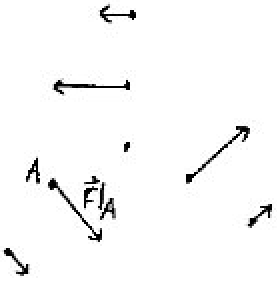
We'll study <u>steady vector fields</u>, in which vectors do not change with time.

<u>Ex</u> Velocity field for a kitchen sink (See 18.3.9, <u>Note 2</u>)

Show some of the vectors, enough to show a pattern.

Use **A** as the initial point for $\vec{F}|_A$.

Sample units:
$\left\langle \frac{m}{sec}, \frac{m}{sec} \right\rangle$



<u>Ex</u> Electromagnetic force fields
Gravitational

Given a scalar multivariable func in Ch.16, what vector field did we construct? VVF

<u>Ex</u> (Ch. 16) Gradient vector field

<u>Ex</u>

$$f(x,y) = x^2 + y^2$$
$$\vec{F}(x,y) = \vec{\nabla} f(x,y) = \langle 2x, 2y \rangle$$

} Ch.16 } Ch.18 (18.3)

We call f a <u>potential function</u> for $\vec{F}$.
-f in Physics so that: $\left.\text{Lower pot.}\right) \xleftarrow{\vec{F} = \vec{\nabla} f} \left(\text{Higher pot.}\right.$

Like the GOP. must be conservative to have potential.

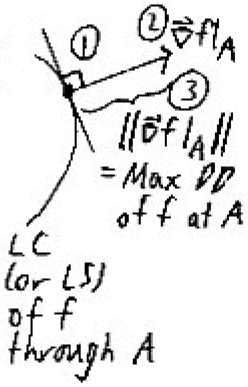
We call $\vec{F}$ a <u>conservative vector field</u>, because $\vec{F} = \vec{\nabla} f$ for some scalar func. f (i.e., $\vec{F}$ has a potential function).

Recall (from 16.6):

① $\underbrace{\vec{\nabla} f|_A}_{= \vec{F}|_A \text{ here}} \perp$ level $\overset{LC}{\text{curve}}$/$\overset{LS}{\text{surface}}$ of the potential f containing A

② $\vec{\nabla} f|_A$ points in direction of max rate of ↗ of f at A.

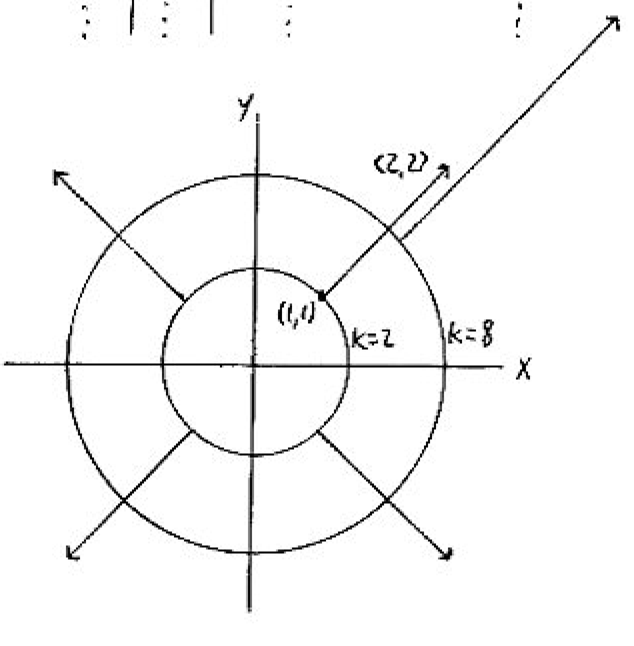
③ Its length, $\|\vec{\nabla} f|_A\|$ is that max rate of ↗.



$\vec{F}|_A$
$\vec{\nabla} f|_A$

Table

A		$\vec{F}(x,y) =$ $\vec{\nabla} f(x,y) =$	(LC thru A) $k =$ $f(x,y) =$
x	y	$\langle 2x, 2y\rangle$	x^2+y^2
1	1	$\langle 2, 2\rangle$	2
-1	1	$\langle -2, 2\rangle$	2
2	2	$\langle 4, 4\rangle$	8
⋮	⋮	⋮	⋮



Ⓑ $\vec{\nabla}$ Operator

del/nabla

$$\vec{\nabla} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \quad \text{(Informal)}$$

$$\underset{\substack{\uparrow \\ \text{scalar} \\ \text{func.}}}{\vec{\nabla} f} = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle \quad \text{gives a VVF (vector field)}$$

} $\mathbb{R}^3$ versions

Marsden uses C^1 instead of "nice." "Smooth" implies derivs non-0, perhaps.

How does $\vec{\nabla}$ operate on VVFs?

(Assume $\vec{F}$ is "nice": components are cont. and have cont. 1st-order PDs where we care.

In 18.7, Swokowski says $\text{rot } \vec{F} = (\overrightarrow{\text{curl}}\, \vec{F}) \cdot \vec{n}$

Ⓒ $\underset{\text{(or rot)}}{\overrightarrow{\text{curl}}}\ \vec{F} = \vec{\nabla} \times \vec{F}$

($\vec{F}$ in $\mathbb{R}^3$) (If $\vec{F} = \langle F_1, F_2 \rangle$ in $\mathbb{R}^2$, write $\langle F_1, F_2, \underset{\nwarrow \text{zero}}{\mathbf{0}} \rangle$)

gives a VVF (vector field)

Ex (#18) If $\vec{F}(x,y,z) = \langle \underbrace{x^3 \ln z}_{M(x,y,z)}, \underbrace{xe^{-y}}_{N(x,y,z)}, \underbrace{-(y^2+2z)}_{P(x,y,z)} \rangle$, find $\overrightarrow{\text{curl}}\, \vec{F}$.

$$\overrightarrow{\text{curl}}\, \vec{F} = \vec{\nabla} \times \vec{F}$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^3 \ln z & xe^{-y} & -(y^2+2z) \end{vmatrix}$$

$$= \left\langle \frac{\partial}{\partial y}[-(y^2+2z)] - \cancel{\frac{\partial}{\partial z}(xe^{-y})}^{\,0}, \right.$$

$$\ominus \left[\cancel{\frac{\partial}{\partial x}[-(y^2+2z)]}^{\,0} - \frac{\partial}{\partial z}(x^3 \ln z) \right],$$

$$\left. \frac{\partial}{\partial x}(xe^{-y}) - \cancel{\frac{\partial}{\partial y}(x^3 \ln z)}^{\,0} \right\rangle$$

$$= \left\langle -2y, x^3 \cdot \frac{1}{z}, e^{-y} \right\rangle$$

$$= \boxed{\left\langle -2y, \frac{x^3}{z}, e^{-y} \right\rangle}$$

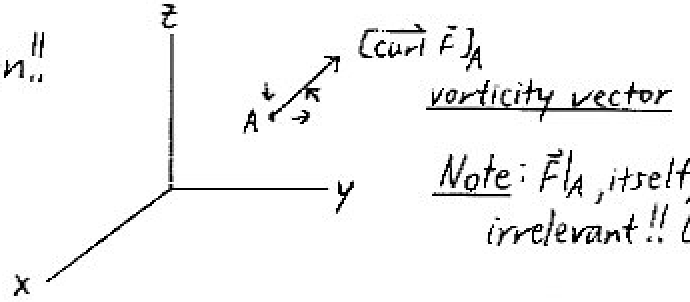
where $\vec{F}$ = velocity VVF

Don't need Cartesian!

(D) Interpreting $[\overrightarrow{\text{curl}}\,\vec{F}]_A$ (in a coordinate-free sense)

Explained on pp. 1013-4 using Stokes's Thm. in 18.7.

"Local" rotation!!



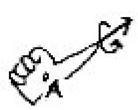
Note: $\vec{F}|_A$, itself, is irrelevant!! (Limit idea)

Direction of $[\overrightarrow{\text{curl}}\,\vec{F}]_A$ indicates the axis of rotation of $\vec{F}$ near A.

which way does $[\overrightarrow{\text{curl}}\,\vec{F}]_A$ point?

rotation is counter-clockwise if look from tip of $\overrightarrow{\text{curl}}$ vector towards the paddlewheel

Right-Hand Rule for $\overrightarrow{\text{curl}}$



curling of fingers indicate "overall rotation" (see ↓) of $\vec{F}$ near A

Idea

Analogous to idea of DDs at A max'ed in direction of $\vec{\nabla} f|_A$

At A, rotate this paddlewheel until its paddles revolve the fastest

See Swok 1016, MIT Munkres Calc II - F9

See HW

$\|[\overrightarrow{\text{curl}}\,\vec{F}]_A\|$ indicates the strength of the rotational effect about near A. It equals twice the angular speed of the paddles about this axis. Sample units: $\frac{\text{radians}}{\text{sec}}$ ← Time unit used for $\vec{F}$

$\overrightarrow{\text{curl}}\,\vec{F} = \vec{0}$ throughout D ⟺ $\vec{F}$ is irrotational in D

Ⓔ $\text{div}\,\vec{F} = \vec{\nabla}\cdot\vec{F}$

divergence gives a scalar function

Ex (#18) If $\vec{F}(x,y,z) = \langle x^3 \ln z,\ xe^{-y},\ -(y^2+2z)\rangle$, find $\text{div}\,\vec{F}$.

$$\text{div}\,\vec{F} = \vec{\nabla}\cdot\vec{F}$$

$$= \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle x^3 \ln z,\ xe^{-y},\ -(y^2+2z)\rangle$$

$$= \frac{\partial}{\partial x}(x^3 \ln z) + \frac{\partial}{\partial y}(xe^{-y}) + \frac{\partial}{\partial z}[-(y^2+2z)]$$

$$= 3x^2 \ln z + xe^{-y}(-1) + (-2)$$

$$= \boxed{3x^2 \ln z - xe^{-y} - 2}$$

Explained using Div. Thm. in 18.6

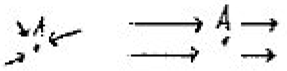
Stewart 1094 Sed ET

Ⓕ Interpreting $[\text{div}\,\vec{F}]_A$ (again, in terms of local behavior near A)

tendency of fluid to diverge from pt. A

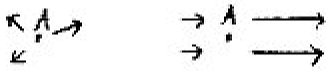
Ex cooling gas - compressible

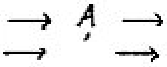
If $[\text{div}\,\vec{F}]_A < 0$, then there is a sink at A.
⊖

Tendency 

alphabetical order ≈ - to +

If $[\text{div}\,\vec{F}]_A > 0$, then there is a source at A.
⊕



If $[\text{div}\,\vec{F}]_A = 0 \Rightarrow$ neither. 

If $\text{div}\,\vec{F} = 0$ throughout $D \Rightarrow \vec{F}$ is divergence free.

incl. water, pretty much

Exs incompressible fluids, solenoidal electromagnetic fields (EM)

See 18.6.1 for units.

⑥ When is a Vector Field, $\vec{F}$, Conservative?

(Assume $\vec{F}$ is "nice.")

Marsden 5ed p.551: In $\mathbb{R}^3$, $\vec{F}$ can be conserv. even if undef. at a finite # of pts. (provided f also undef. there). Exceptional pts. not allowed in $\mathbb{R}^2$?

$\Leftrightarrow$ $\vec{F} = \vec{\nabla} f$ for some scalar func. f

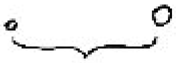
$\Leftrightarrow$ $\overrightarrow{\text{curl}}\ \vec{F} = \vec{0}$ (if $\vec{F}$ nice in $\mathbb{R}^3$)

(You'll prove $\Rightarrow$ in $\underset{\#22}{\text{HW}}$! Converse ($\Leftarrow$) proven later in 18.7 on Stokes's Thm.)

Larson: This is related to conservation of energy.
For a particle moving in a conservative force field,
the sum of its $\underset{\text{due to motion}}{\text{kinetic energy}}$
and its $\underset{\text{due to position}}{\text{potential energy}}$
is constant.

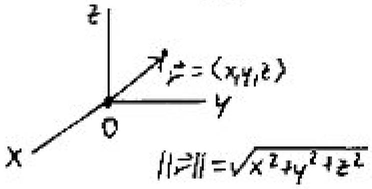
Ⓗ Inverse Square Fields are Conservative

<u>Ex</u> Gravity, Electric force (Coulomb's Law)



The magnitude of the force between these is inversely proportional to the square of the distance between them. Let one be at 0.

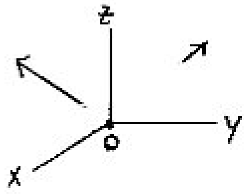
2× distance ⇒ $\frac{1}{4}$ of force
3 $\frac{1}{9}$

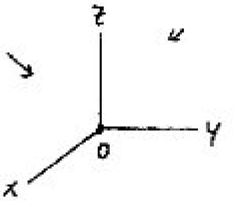


$$\vec{F}(x,y,z) = \underbrace{\left(\frac{c}{\|\vec{r}\|^2}\right)}_{\text{ensures that } \|\vec{F}(x,y,z)\| = \frac{|c|}{\boxed{\|\vec{r}\|^2}}}\underbrace{\left(\frac{\vec{r}}{\|\vec{r}\|}\right)}_{\text{unit vector in direction of } \vec{r}}$$ for some constant of proportionality, c.

$$= \boxed{\frac{c\vec{r}}{\|\vec{r}\|^3}}$$, provided $\vec{r} \neq \vec{0}$

If $c > 0$, repulsion (away from 0) If $c < 0$, attraction (towards 0)





These vectors are ⊥ to $\underbrace{\text{spheres centered at } 0.}_{\text{level surfaces of potential}}$

In $\mathbb{R}^3$, so OK that $\vec{0}$ is an exceptional pt.

$\vec{F}$ is conserv.: $\vec{F} = \vec{\nabla} f$, where $f(x,y,z) = -\frac{c}{\|\vec{r}\|}$ (✓ this !!)

Ⓘ Interesting IDs (in $\mathbb{R}^3$)

Stewart 5edET. pp. 1091, 7: If $\vec{F}$ comps, f have cont. 2nd-order PDs

$\text{div}(\overrightarrow{\text{curl}}\,\vec{F}) = 0$
$\overrightarrow{\text{curl}}\,(\underbrace{\vec{\nabla} f}_{\text{is conserv., remember?}}) = \vec{0}$ (see 18.1.3 figure)

div, $\overrightarrow{\text{curl}}$ critical to Maxwell's Laws in EM !!

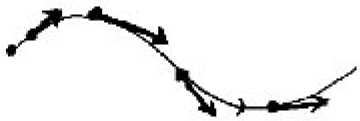
Marsden 5ed, 290-2

Ⓙ Flow Lines

(or Streamlines or Integral Curves)

If $\vec{F}$ is a velocity field, then a particle placed in the field will trace out a flow line.

(See 15.2.2)



Flow line carries "invisible" speed info.

If that's the path we want, how much does $\vec{F}$ help us?
What if we want another path?
Even if there's a flow line from pt. A to pt. B, can we do better?

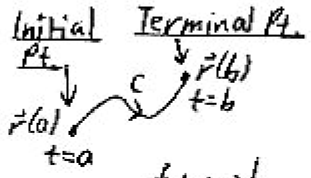
We'll discuss Work in 18.2 Ⓔ.

18.2: LINE (PATH) ∫s

We'll do 2D, but this extends to 3D easily.

Ⓐ Smooth Curves

$\vec{r}(t) = \langle x(t), y(t) \rangle$ gives a smooth parameterization of a curve, C, $\underbrace{\text{on } [a,b]}_{\text{i.e., when } a \le t \le b}$



$t: a \to b$

$\Leftrightarrow \vec{r}\,'(t) = \left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle$, a tangent VVF, is

① cont. on $[a,b]$, and
② never $\vec{0}$ on (a,b)

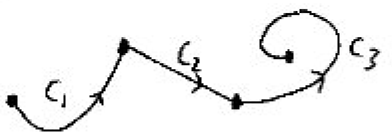
Then, C is a smooth curve with no breaks, corners, or cusps.

\ /   V   Y

$\vec{r}$ can't backtrack.

Ⓑ Piecewise-Smooth $\overset{\text{my notation}}{\downarrow}$ (ps) Curves

can be partitioned into a finite # of smooth curves.



$C = C_1 \underset{\substack{\uparrow \\ \text{union}}}{\cup} C_2 \cup C_3$

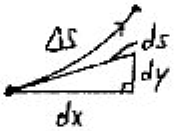
18.2.2

Ⓒ Mass, m, of a ps Wire, C

Recall (15.1)

ds = differential of arc length "s"

In Δt time,



$ds \approx \Delta s$

In 3D

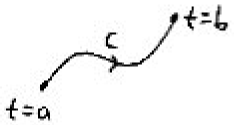
$\sqrt{\dots + (dz)^2}$ $\quad = \sqrt{(dx)^2 + (dy)^2}$ (Informal)

$\sqrt{\dots + (\frac{dz}{dt})^2}$ $\quad = \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}\,dt$ (If $t\nearrow$; otherwise, "$|dt|$")

$$= \|\vec{r}\,'(t)\|\,dt \quad \text{or} \quad \underbrace{\underbrace{\|\vec{v}(t)\|}_{\text{speed}}\underbrace{dt}_{\Delta t = \text{change in time}}}_{\text{distance covered}}$$

$$\text{Arc Length of } C = \int_C ds$$

$$= \int_a^b \sqrt{(\tfrac{dx}{dt})^2 + (\tfrac{dy}{dt})^2}\,dt$$



Now

Let $\delta(x,y)$ be the linear mass density of a wire at (x,y).

(Assume density is constant in a cross-section.)

Idea

Break C into tiny arcs.

$P_k(x_k, y_k)$ is *any* point on C_k.

$\delta \approx$ constant on C_k. ← tiny!
C_k: cover length Δs_k in time Δt_k

$$\text{Mass of } C_k \approx (\text{Density at } P_k)(\text{Arc length of } C_k) = [\delta(x_k, y_k)][\Delta s_k]$$

$$\boxed{\text{Mass, } m, \text{ of } C = \int_C \delta(x,y)\, ds}$$

Why?

$$\int_C \delta(x,y)\, ds = \lim_{\underbrace{\|P\| \to 0}_{\text{largest } \Delta t_k}} \overset{\text{Riemann sum}}{\underbrace{\sum_k \underbrace{\delta(x_k, y_k)\Delta s_k}_{\text{Mass of } C_k}}_{\text{Approx. for } m}}$$

This is an example of a...

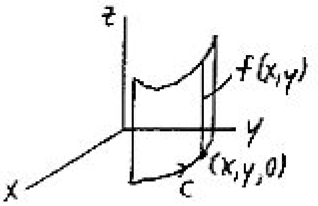
Stewart 5e, ET 1062: Invented in early 19c to study forces, fluid flow, EM

Ⓓ Line (Path) Integral

$$\boxed{\int_C f(x,y)\,ds}$$

If $f(x,y) = 1 \Rightarrow$ Arc Length of C
If $f(x,y) = \delta(x,y) \Rightarrow$ Mass of C

Lateral Surface Area ("Wall")



if $f(x,y) \geq 0$ along C

DETAILS...

Note 1: (We assume C is ps, and f is cont. on a region containing C.)

Note 2: We get the same value for $\int_C f(x,y)\,ds$, regardless of how we parameterize C. (if smooth)

Even the orientation doesn't matter:

$$\int_C f(x,y)\,\textcircled{ds} = \int_{-C} f(x,y)\,\textcircled{ds}$$

⇐ Expand using $|dt|$

Always true: $\Delta s_k \geq 0$, mass ≥ 0
We flip sign if we had $\textcircled{dx}$, $\textcircled{dy}$, ...

(e.g., $\int_C f(t)\,dt = -\int_C f(t)\,dt$
$t: a \to b$ $t: b \to a$ for otherwise same param.)

If $\Delta S_k = 0$
$\Rightarrow \vec{r}\,' = 0$ on C there, lose "smooth"

Stewart 5e, ET 1067: → Remember $|dt|$ from ds

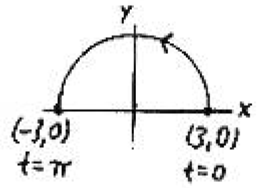
Like Ex 3

<u>Ex</u> Find the mass of a wire C if the density at $P(x,y)$ is directly proportional to its distance from the x-axis, and C is parameterized by $x=-3\cos t$, $y=3\sin t$; $0 \leq t \leq \pi$.
(Assume $t: 0 \to \pi$; $t \nearrow$ consistently w/orientation)

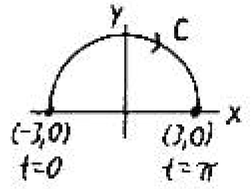
<u>Sol'n</u>

<u>What is C? (Optional!)</u>

If we had $\begin{cases} x = 3\cos t \\ y = 3\sin t \end{cases} \Rightarrow x^2+y^2=9$



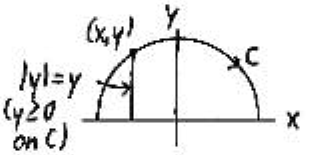
Here, $\begin{cases} x = \ominus 3\cos t \\ y = 3\sin t \end{cases}$



Good: No overlapping.

<u>What is δ(x,y)?</u>

$\delta(x,y) = ky$



<u>What is m?</u>

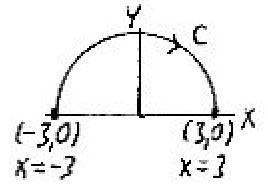
$$m = \int_C \delta(x,y)\, ds$$
$$= \int_0^{\pi} ky \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$
$$= \int_0^{\pi} k(3\sin t) \underbrace{\sqrt{(3\sin t)^2 + (3\cos t)^2}}_{=\sqrt{9\sin^2 t + 9\cos^2 t} = \sqrt{9(1)} = ③}\, dt$$

$x=-3\cos t$, $y=3\sin t$
$\frac{dx}{dt} = 3\sin t$, $\frac{dy}{dt} = 3\cos t$

$$= 9k \left[-\cos t\right]_0^{\pi}$$
$$= 9k \left(\underbrace{-\underbrace{\cos \pi}_{(-1)}}_{1} - \underbrace{(-\underbrace{\cos 0}_{1})}_{(-1)} \right)$$
$$= \boxed{18k}$$

If C lies on the graph of $y=f(x)$; $a\le x\le b \Rightarrow$
let $x=t$, $y=f(t)$; $a\le t\le b$. (Similarly for $x=f(y)$.)

Redo Previous Ex (SKIPPED IN CLASS?)



$$y=\sqrt{9-x^2}\ ,\ -3\le x\le 3$$

$$x=t \quad ; \quad y=\sqrt{9-t^2} \text{ or } (9-t^2)^{1/2}$$

$$\frac{dx}{dt}=1 \quad ; \quad \frac{dy}{dt}=\frac{1}{\cancel{2}}(9-t^2)^{-1/2}(-\cancel{2}t)$$

$$=-\frac{t}{\sqrt{9-t^2}}$$

$$-3\le t\le 3 \quad (t: -3\to 3)$$

$$m=\int_C \delta(x,y)\,ds$$

$$=\underbrace{\int_{-3}^{3}}_{} ky\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$$

(We basically could have used dx.)

or $2\int_0^3$ by sym. of C, $\delta(x,y)=ky$ even in x
about $x=0$

$$=2\int_0^3 k\sqrt{9-t^2}\sqrt{(1)^2+\left(-\frac{t}{\sqrt{9-t^2}}\right)^2}\,dt$$

$$=2\int_0^3 k\sqrt{9-t^2}\sqrt{1+\frac{t^2}{9-t^2}}\,dt$$

$$=2\int_0^3 k\sqrt{(9-t^2)\left(1+\frac{t^2}{9-t^2}\right)}\,dt$$

$$=2\int_0^3 k\sqrt{9\cancel{-t^2+t^2}}\,dt$$

$$=2\int_0^3 3k\,dt$$

$$=6k\int_0^3 dt$$

$$=6k\,[t]_0^3$$

$$=6k(3)$$

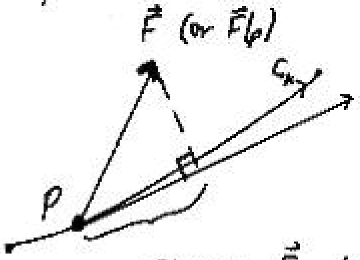
$$=\boxed{18k}$$

Ⓔ Line $\int$ of a Vector Field, $\vec{F}$

Let W = the work done by $\vec{F}$ on a particle moving along C [in the direction of orientation].

C_k, a tiny arc on C:

Normal component has no impact on the particle



$\text{comp}_{\vec{T}}\,\vec{F}$, the tangential component of $\vec{F}$

= ± magnitude of force acting in the direction of $\vec{T}$

$$= \frac{\vec{F}\cdot\vec{T}}{\cancelto{1}{\|\vec{T}\|}}$$

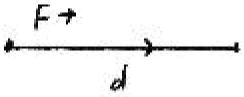
$$= \vec{F}\cdot\vec{T}$$

This depends on $\vec{F}$ and C, but not the particle's speed along C.

(We're measuring the impact that $\vec{F}$ has on the particle as it moves along C.)

Recall Calc II

If F is a constant scalar force, then $W = Fd$ here:



<u>Now</u> C curvy, $\vec{F}$ nonconstant

On tiny C_k, $\vec{F}\cdot\vec{T} \approx$ constant.

Work done along $C_k \approx \underbrace{(\vec{F}\cdot\vec{T})}_{\text{“force impact”}}\ \underbrace{\Delta s_k}_{\text{arc length of } C_k}$

$$\boxed{W = \int_C \underbrace{\vec{F}\cdot\vec{T}}_{\text{scalar function: special case of } f(x,y)}\, ds}$$

Stewart 5ed, ET, 1070

<u>Note</u> $\int_C \vec{F}\cdot\vec{T}\, ds = \ominus \int_{-C} \vec{F}\cdot\underset{\substack{\uparrow\\ \text{actually } -(\text{old } \vec{T})}}{\vec{T}}\, ds$

Orientation matters!
Speed along C
still doesn't.

$$= \int_C \vec{F}\cdot \underbrace{\frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|}}_{=\vec{T}(t)}\ \underbrace{\|\vec{r}\,'(t)\|\, dt}_{=ds}$$

← if t↗, but formula still OK if t↓ (!!)
(see review notes)

$$\boxed{W = \int_C \vec{F}\cdot \underbrace{\vec{r}\,'(t)\, dt}_{\substack{=\frac{d\vec{r}}{dt}dt\\ =\text{“}d\vec{r}\text{”}}}}$$

Maybe best form if $\vec{F}, \vec{r}$ given in terms of t. ($\vec{F}$ only known for points along C.)

$$\boxed{W = \int_C \vec{F}\cdot d\vec{r}}$$

$\vec{F}$ = ⟨M(x,y), N(x,y)⟩, cont. in a region containing C

$d\vec{r}$: Think: ⟨dx, dy⟩

$$\boxed{W = \int_C M\, dx + N\, dy}$$

↖( ) often omitted ↗

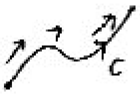
<u>Differential Form.</u>

We'll use this a lot!

In 3D, $W = \int_C M\,dx + N\,dy + P\,dz$

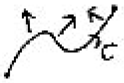
if $\vec{F}(x,y,z) = \langle M(x,y,z), N(x,y,z), P(x,y,z)\rangle$

Idea



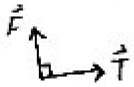
$W \gg 0$ The force is with you!

$\vec{F} \cdot \vec{T} > 0$



$W = 0$ No help/harm.

$\vec{F} \cdot \vec{T} = 0$

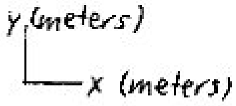


$W \ll 0$ Forces conspiring against you!

$\vec{F} \cdot \vec{T} < 0$

Swok 6.6
Also: ft.-lbs.

Units If $\vec{F}$ lengths in Newtons,



$\Rightarrow$ W in Newton-meters, or joules (J)

1 joule is the force needed to accelerate a 1 kg mass by 1 $\frac{m}{sec^2}$.

18.2.10

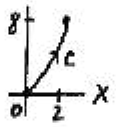
Like #4

<u>Ex</u> $\vec{F}(x,y) = \langle \underbrace{y}_{M(x,y)}, \underbrace{x+y}_{N(x,y)} \rangle$.

C is the graph of $y = x^2 + 2x$ directed from $(0,0)$ to $(2,8)$.
Find the work, W, done by $\vec{F}$ on a particle moving along C.

<u>Sol'n</u>

Draw C (Optional?)



$C: y = \underbrace{x^2 + 2x}_{f(x)}\ ;\ x: 0 \to 2$ (x is our parameter.)

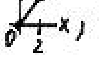
(Also nice: $x = f(y)\ ;\ y: a \to b$)

Use Differential Form:

$$W = \int_C M\,dx + N\,dy$$
$$= \int_C y\,dx + (x+y)\,dy$$

$$\left.\begin{aligned} y &= x^2 + 2x \\ dy &= (2x+2)\,dx \end{aligned}\right\} \left(\text{Write } \begin{aligned} x &= \dots \\ dx &= \dots \end{aligned} \text{ if } x = f(y)\right)$$

$$= \int_0^2 \underbrace{(x^2+2x)}_{y}\,dx + [x + \underbrace{(x^2+2x)}_{y}]\underbrace{(2x+2)\,dx}_{dy}$$
$$= \int_0^2 [x^2 + 2x + (x^2+3x)(2x+2)]\,dx$$
$$\vdots$$
$$= \boxed{48}$$

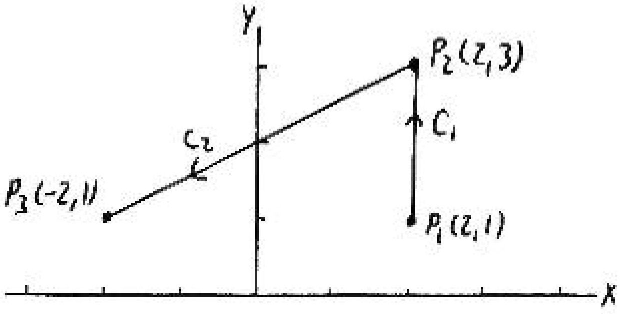
<u>Note</u>: If , then $x: 2 \to 0$, and $W = \int_2^0 \cdots = \boxed{-48}$. Makes sense!

OK if x ↓ in direction of motion.

$\vec{F}$ hurts us commensurately w/ how it helped us before.

<u>Ex</u> $\vec{F}(x,y) = \langle xy^2, e^{2y} \rangle$.

$C = C_1 \cup C_2$:



Find W.

<u>Method 1</u> (Know both methods!): Use t as a parameter.

<u>ⓐ Parameterize C_1, C_2</u>

Ⓒ₁ $C_1: \begin{cases} x = 2 \\ y = t \end{cases} \Rightarrow \begin{cases} dx = 0 \\ dy = dt \end{cases}$

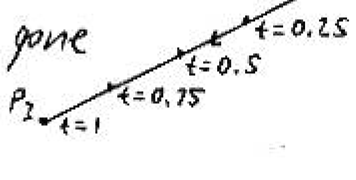
$t: 1 \to 3$

Ⓒ₂ Initial Point: $P_2(2,3)$
Displacement Vector: $\overrightarrow{P_2P_3} = \langle -2-2, 1-3 \rangle$
$= \langle -4, -2 \rangle$

$C_2: \begin{cases} x = 2 - 4t \\ y = 3 - 2t \end{cases} \Rightarrow \begin{cases} dx = -4\,dt \\ dy = -2\,dt \end{cases}$

$t: \underset{\substack{\Downarrow \\ P_2}}{0} \to \underset{\substack{\Downarrow \\ P_3}}{1}$

t = fraction of the way you've gone from P_2 to P_3



(b) <u>Find W</u>

$$W = \int_{C_1} M\,dx + N\,dy \qquad + \quad \int_{C_2} M\,dx + N\,dy$$

$$= \int_{C_1} xy^2 dx + e^{2y} dy \qquad + \quad \int_{C_2} xy^2 dx + e^{2y} dy$$

$$= \int_1^3 \overset{\leftarrow t\text{-values} \leftarrow}{\underbrace{(2)(t)^2(0)}_{=0} + e^{2(t)} dt}$$

$$+ \int_0^1 (2-4t)(3-2t)^2(-4)dt + e^{2(3-2t)}(-2)dt$$

$$\approx 198 - 209$$

$$= \boxed{-11}$$

<u>Method 2</u>: Use x and/or y as parameters.

(C_1) C_1: $x = 2 \Rightarrow dx = 0$

$y: 1 \to 3$

(C_2) Point: $(2,3)$

Slope $= \frac{1-3}{-2-2} = \frac{1}{2}$

Pt.-Slope form: $y - 3 = \frac{1}{2}(x-2)$

$\Rightarrow$ Slope-Int. form: $y = \frac{1}{2}x + 2$

C_2: $x: 2 \to -2$

$y = \frac{1}{2}x + 2 \Rightarrow dy = \frac{1}{2}dx$

$$W = \int_{C_1} M\,dx + N\,dy \qquad + \quad \int_{C_2} M\,dx + N\,dy$$

$$= \int_{C_1} xy^2 dx + e^{2y} dy \qquad + \quad \int_{C_2} xy^2 dx + e^{2y} dy$$

$$= \int_1^3 \overset{\leftarrow y\text{-values} \leftarrow}{\underbrace{(2)y^2(0)}_{=0} + e^{2y} dy} \;+ \int_2^{-2} \overset{\leftarrow x\text{-values} \leftarrow}{x\left(\tfrac{1}{2}x + 2\right)^2 dx + e^{2(\frac{1}{2}x+2)} \cdot \tfrac{1}{2}dx}$$

$\vdots$

$\approx \boxed{-11}$; same as for Method 1

18.3: INDEPENDENCE OF PATH (IP)

Ⓐ Assumptions

C is ps.

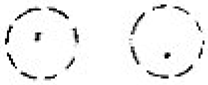
D is a simply connected open region containing C.

open: excludes boundary

connected: "in one piece": (any pair of points in D can be joined by a ps curve in D)

simply: Don't need until Ⓔ

NO



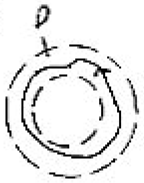
No Michigans

D has no holes (if in $\mathbb{R}^2$).
i.e., No simple closed curve in D encloses points not in D.

simple closed curve: $\vec{r}(a) = \vec{r}(b)$, and the only self-intersection point is there.

In $\mathbb{R}^3$, can have finite # of exceptional pts. where $\vec{F}$, f undef. for some of our Thms.

NO



Simply connected Ds in $\mathbb{R}^3$ discussed in 18.7 (pp. 1017-8). We'll extend Ⓔ into 3D then, anyway.

$\vec{F}$ is cont. in D.

(B) Indep of Path (IP)

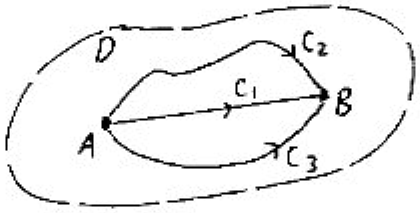
Find $\int_C \vec{F} \cdot d\vec{r}$.

What if C is hard to parameterize? 

Can we use  , instead?

We have $\underset{\text{indep. of path}}{\overset{(IP)}{}}$ for $\vec{F}$ in $D \Longleftrightarrow$

For <u>any</u> pair of points A and B in D, $\int_C \vec{F} \cdot d\vec{r}$ yields the same #, <u>regardless</u> of which ps curve C in D from A to B we use.



We can then say: $\int_A^B \vec{F} \cdot d\vec{r}$.

© Showing Indep.$^{(IP)}$ of Path for $\vec{F}$ in D by Finding a Potential Function, f

In D:

$$\boxed{\begin{aligned} \int_C \vec{F}\cdot d\vec{r} \text{ is } \underset{(IP)}{\text{indep.}} \text{ of path} &\iff \vec{F} \text{ is conservative} \\ &\iff \vec{F} = \vec{\nabla} f \text{ for some} \\ &\qquad \text{scalar potential func., } f \end{aligned}}$$

(Note that exists throughout D, though, in $\mathbb{R}^3$, we may have a finite # of exceptional pts. where $\vec{F}, f$ undefined. See 18.3.9 for an Ex in $\mathbb{R}^2$ where a suspected f doesn't work out.

Proof pp. 982-4

Note: D need not be simply connected.
$\nwarrow$Need

Like #4

Ex $\vec{F}(x,y) = \langle 2xe^{2y} + 4y^3,\ 2x^2e^{2y} + 12xy^2 - 2y \rangle$.
Show that $\int_C \vec{F}\cdot d\vec{r}$ is $\underset{(IP)}{\text{indep. of}}$ path throughout $\mathbb{R}^2$.

Sol'n

Find a potential, f, such that

$$\vec{\nabla} f = \vec{F}$$

$$\langle f_x(x,y), f_y(x,y)\rangle = \langle 2xe^{2y} + 4y^3,\ \underbrace{2x^2e^{2y} + 12xy^2 - 2y}_{✪} \rangle$$

Partially $\int f_x$ wrt x $\quad f_x \nearrow f \quad$ ((Could $f \nwarrow_{f_y} \int \text{wrt } y$)

$$\int f_x(x,y)\,dx = \int (2xe^{2y} + 4y^3)\,dx$$

$$f(x,y) = 2e^{2y}\left(\frac{x^2}{2}\right) + 4y^3x + \underbrace{g(y)}_{D_x[g(y)]=0}$$

$$f(x,y) = x^2e^{2y} + 4xy^3 + g(y) \quad (☆☆☆)$$

D_y both sides $\quad f \searrow f_y$

$$f_y(x,y) = D_y[x^2e^{2y} + 4xy^3 + g(y)]$$
$$= x^2(2e^{2y}) + 4x(3y^2) + g'(y)$$
$$= 2x^2e^{2y} + 12xy^2 + \textcircled{g'(y)}$$

Compare with ☆ $\Rightarrow f_y(x,y) = 2x^2e^{2y} + 12xy^2 \underset{☆☆}{\textcircled{-2y}}$

$\int$ ☆☆ wrt y to get g(y)

$$\int g'(y)\,dy = \int -2y\,dy$$
$$g(y) = -y^2 + \underset{\text{"C" already used}}{K}$$

Write out f(x,y). Use ☆☆☆.

In fact, this = $\int f_y(x,y)\,dy$

$$\boxed{f(x,y) = x^2e^{2y} + 4xy^3 - y^2 + K}$$ Can D_x, D_y to ✓.

Only Larson, as far as I know

Alternative Method (Don't use, though)

$$\int f_x(x,y)\,dy = \cdots = x^2e^{2y} + 4xy^3 + g(y)$$
$$\int f_y(x,y)\,dy = \cdots = x^2e^{2y} + 4xy^3 - y^2 + h(x)$$

Form a guess: $f(x,y) = x^2e^{2y} + 4xy^3 - y^2 + K$

D_x, D_y to ✓.

Ex Find a potential, f, for $\vec{F}(x,y,z) = \langle \underbrace{8x}_{f_x}, \underbrace{-9z}_{f_y}, \underbrace{-9y + 3z^2}_{f_z} \rangle$.

Sol'n

$f_x : 8x$

$$f : \int 8x\,dx = 4x^2 + \underbrace{g(y,z)}_{\text{Could have } y,\, z,\, \text{both, or neither}}$$

$f_y : g_y(y,z)$

$f_y : -9z$

$\Downarrow$

$$\begin{aligned} g_y(y,z) &= -9z \\ g(y,z) &= \int -9z\,dy \\ &= -9zy + h(z) \end{aligned}$$

$f : 4x^2 - 9yz + h(z)$ ✪

$f_z : -9y + h'(z)$

$f_z : -9y + 3z^2$

$\Downarrow$

$$\begin{aligned} -9y + h'(z) &= -9y + 3z^2 \\ h'(z) &= 3z^2 \\ h(z) &= \int 3z^2\,dz \\ h(z) &= z^3 + K \end{aligned}$$

✪ $\Rightarrow$ $\boxed{f(x,y,z) = 4x^2 - 9yz + z^3 + K}$

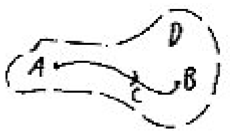
Alternative Method (Don't use!)

$$\left.\begin{aligned} \int f_x(x,y,z)\,dx &= 4x^2 + g(y,z) \\ \int f_y(x,y,z)\,dy &= -9yz + h(x,z) \\ \int f_z(x,y,z)\,dz &= -9yz + z^3 + \ell(x,y) \end{aligned}\right\} \Rightarrow \text{Guess: } f(x,y,z) = 4x^2 - 9yz + z^3 + K$$

D_x, D_y, D_z to ✓.

Stewart calls

Ⓓ Fundamental Thm. for Line $\int$s (FTLI) - extends FTC from Calc I



If $\vec{F}$ is conservative in D with potential f ($\vec{F} = \vec{\nabla} f$), then

$$\int_C \vec{F}\cdot d\vec{r} = \int_A^B \vec{F}\cdot d\vec{r} \quad \text{(by Indep.}\overset{\text{(IP)}}{\text{ of }}\text{Path for } \vec{F} \text{ in } D\text{)}$$

$$= [f(x,y)]_A^B \quad \text{or} \quad [f(x,y,z)]_A^B$$

$$= f\big|_B - f\big|_A$$

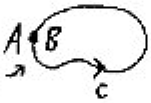
Physical Interpretation

The work done by a conservative force field $\vec{F}$ along any path C from A to B

= The difference in potentials between A and B.

What if $A=B$?

closed curve



$$\int_C \vec{F}\cdot d\vec{r} = \int_A^A \vec{F}\cdot d\vec{r}$$

$$= f\big|_A - f\big|_A$$

$$= 0$$

Doesn't matter where you start. If do 1 full circuit ↺, same net effect.

↩ : $-0=0$

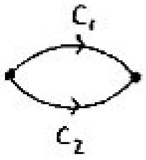
"Simple" helps. Orientation unclear for

where do we go when we hit this pt.?

True "Converse":

If $\int_C \vec{F}\cdot d\vec{r} = 0$ for <u>every</u> simple closed curve C in D, then $\vec{F}$ is conservative, and we have indep. of path for $\vec{F}$ in D.
(IP)

Why?



If $\int_C \vec{F}\cdot d\vec{r} = 0$

$\circlearrowleft$

$\Rightarrow \int_{C_1} \text{〃} + \int_{-C_2} \text{〃} = 0$

$\Rightarrow \int_{C_1} \text{〃} - \int_{C_2} \text{〃} = 0$

$\Rightarrow \int_{C_1} \text{〃} = \int_{C_2} \text{〃}$

Ex From Ⓒ: $\vec{F}(x,y) = \langle 2xe^{2y} + 4y^3,\ 2x^2e^{2y} + 12xy^2 - 2y\rangle$.
($\nwarrow$conservative in $\mathbb{R}^2$).

Evaluate $\int_{(1,2)}^{(3,4)} \vec{F}\cdot d\vec{r}$.

Method 1: Use FTLI.

$= \left[f(x,y)\right]_{(1,2)}^{(3,4)}$

$= \left[x^2e^{2y} + 4xy^3 - y^2\right]_{(1,2)}^{(3,4)}$ from Ⓒ

$\uparrow$ Don't need "+K."

752 + 9e^8

-[28+e^4]

$= \left[(3)^2e^{2(4)} + 4(3)(4)^3 - (4)^2\right]$
$- \left[(1)^2e^{2(2)} + 4(1)(2)^3 - (2)^2\right]$

$= \boxed{9e^8 - e^4 + 724}$

whether we

Method 2: Use Ⓔ$\Rightarrow$, 18.2 method for "easy" C $\nearrow$

Ⓔ Showing $\overset{(IP)}{\text{Indep.}}$ of Path for $\vec F$ in D (Method 2)

$\overbrace{\text{If } \vec F = \langle M,N\rangle}^{\text{in } \mathbb{R}^2}$, and $\overbrace{M,N \text{ have cont. } 1^{st} \text{ PDs}}^{\vec F \text{ is ``nice''}}$ Ⓐ on D,
and if D is $\underset{\text{no holes}}{\underline{\text{simply}}}$ connected, then

#19: $\vec F = \langle M,N,P\rangle$
$M_y = N_x$, $M_z = P_x$,
$N_z = P_y$

vs. $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$
M_y order better for 18.4 →

$$\boxed{\begin{array}{l} \int_C \vec F \cdot d\vec r = \int M\,dx + N\,dy \text{ is indep. of path (IP)} \\ \iff \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y} \quad \text{for } \underline{\text{all}}\ (x,y) \text{ in } D \end{array}}$$

Proof Idea

$(\Rightarrow)$ IP $\Rightarrow$ There exists f:

$$\begin{tikzcd} & f \arrow[dl, dash, "D_x"'] \arrow[dr, dash, "D_y"] & \\ \vec F:\ M \arrow[dr, dash, "D_y"'] & & N \arrow[dl, dash, "D_x"] \\ & M_y = N_x & \end{tikzcd}$$

$\uparrow$ because $f_{xy} = f_{yx}$ if Ⓐ

$(\Leftarrow)$ Hard! Requires that D be $\underline{\text{simply}}$ connected.

Ex from Ⓒ: $\vec F(x,y) = \langle \underbrace{2xe^{2y} + 4y^3}_{M(x,y)}, \underbrace{2x^2e^{2y} + 12xy^2 - 2y}_{N(x,y)} \rangle$.
Assume 1^{st} PDs are cont. on $\mathbb{R}^2$.

Show that $\int_C \vec F \cdot d\vec r$ is indep. of path (IP) throughout $\mathbb{R}^2$.
(In Ⓒ, we did this by finding a potential.)

Sol'n

$\frac{\partial N}{\partial x} = 4xe^{2y} + 12y^2$
for all (x,y) in $\mathbb{R}^2$

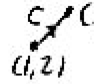
$$\frac{\partial M}{\partial y} = 2x(2e^{2y}) + 12y^2$$
$$= 4xe^{2y} + 12y^2$$ 〃 〃

$\Rightarrow \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$ 〃 〃

$\Rightarrow \int_C \vec F \cdot d\vec r$ is IP throughout $\mathbb{R}^2$.

If C in D;
here, $D = \mathbb{R}^2$.

Note: To compute $\int_{(1,2)}^{(3,4)} \vec F \cdot d\vec r$, we can use 18.2 on



$\begin{cases} x = 1+2t \\ y = 2+2t \\ t: 0 \to 1 \end{cases}$

Note 1

C can't pass over a "hole."

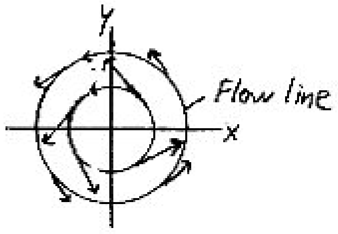
If $\frac{\partial N}{\partial x} \neq \frac{\partial M}{\partial y}$ at <u>any</u> (x,y) in D, then we do not have IP for $\vec{F}$ in D.

Note 2 (<u>How to Ace the Rest of Calculus</u>, pp. 234-5) (Skim in class)

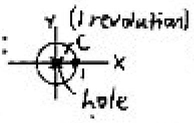
Like #27 $\langle -, + \rangle$

$\vec{F}$ is an approx. "kitchen sink" field. Vectors longer as $\|\vec{r}\|$ goes from $1 \to 0$ $\uparrow$ $\vec{F}$ undef. This is "approximately" a sink; water doesn't move in!

If $\vec{F}(x,y) = \left\langle \frac{y}{x^2+y^2}, -\frac{x}{x^2+y^2} \right\rangle$, then



$\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$, <u>except</u> at $(0,0)$.

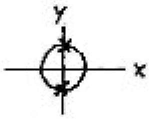
Turns out for: 

No <u>simply</u> connected region D contains C.

$$\int_C \vec{F} \cdot d\vec{r} \underset{\substack{\uparrow \\ \text{Using 18.2.}}}{=} 2\pi \neq 0$$

$\vec{F}$ is not conservative on any region containing C.

If you attempt to construct a potential, f, you may get $-\tan^{-1}\left(\frac{y}{x}\right)$, but observe that this is undefined when $x=0$.



Note 3

In 18.7, we will extend Ⓔ to 3D.

(Typed in review:)

Ⓔ When is $\vec{F}$ Conservative in $\mathbb{R}^2$? Equivalent Statements:

In a connected region D (in which $\vec{F}$ is cont.)...
"in one piece"

① $\vec{F}$ is conservative
(i.e., $\vec{F} = \vec{\nabla} f$ for some scalar potential func. f)

② We have IP : $\int_C \vec{F} \cdot d\vec{r}$

③ $\int_C \vec{F} \cdot d\vec{r} = 0$ for every simple closed curve C in D

(4a) $\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$ throughout D

if $\vec{F} = \langle M, N \rangle$ is "nice."

Note: $\int_C \vec{F} \cdot d\vec{r} = \int_C M\,dx + N\,dy$

If we start with (4a), then we require that D be simply connected.
no holes (in $\mathbb{R}^2$)

When we discuss the $\mathbb{R}^3$ case in 18.7, we will replace (4a) with (4b).

18.4: GREEN'S THEOREM

George Green (1793-1841) was an English mathematical physicist who published this theorem in an EM paper in 1828. Self-taught!

Ⓐ Prelims

In $\mathbb{R}^2$:

Let C be a ps simple closed curve that is the boundary of $R = C \cup$ interior of C.

D is an open region containing R.

C is boundary of R: $C = \partial R$

Let $\vec{F} = \langle M, N \rangle$, where M, N are "nice" throughout D.

i.e., are cont. and have cont. 1st PDs

$\oint_C \vec{F} \cdot d\vec{r}$

$\int$ along C once in the positive direction

R always on the left

counterclockwise unless hole; see ©

$\oint$ also used

Ⓑ Green's Thm

Green's Thm. an extension of FTC.

$$\boxed{\oint_C \vec{F}\cdot d\vec{r} = \oint_C M\,dx + N\,dy = \iint_R \underbrace{\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)}_{} dA}$$

$=0$ if $\vec{F}$ is conservative throughout D
(not "if and only if"; could be 0 even if $\vec{F}$ isn't)

Proof Note (FTC)
Fundamental Thm. of Calculus is used to equate a $\iint$ involving partials with a "1-D" $\int$ along a boundary.

Ⓒ Area of R

Note If you use $\vec{F}(x,y) = \langle \overset{M}{-y}, \overset{N}{x} \rangle$, then

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_R (1-(-1))\,dA$$

$$= 2\iint_R dA$$

$$= 2\cdot(\text{Area of } R)$$

$$\Rightarrow \boxed{\text{Area of } R = \frac{1}{2}\oint_C -y\,dx + x\,dy}$$

Often easier to use than: Area $= \oint_C x\,dy$ or $\oint_C -y\,dx$

How to Ace: You can judge a book by its cover!

Larson 5e, ET 1023

Try this: Show that the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is πab. $(a>0, b>0)$

Hint: See my 13.1 Notes.

May be easier to remember:

$$\text{Area of } R = \frac{1}{2}\oint_C x\,dy - y\,dx$$

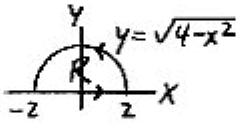
Ⓓ Ex (#6)

Evaluate $\oint_C y^2\,dx + x^2\,dy$, where

C is the boundary of the region bounded by the semicircle $y = \sqrt{4-x^2}$ and the x-axis.

Sol'n

Draw C



$$\oint_C \underbrace{y^2}_{M}\,dx + \underbrace{x^2}_{N}\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA$$

(both nice)

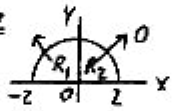
$$= \iint_R (2x - 2y)\,dA$$

In Cartesian coords.,

$$= \underbrace{\int_{x=-2}^{x=2}}\int_{y=0}^{y=\sqrt{4-x^2}} (2x-2y)\,dy\,dx$$

<u>NOT</u> $\nRightarrow 2\int_0^2$, because $\overbrace{\text{not even}}$ in x.

Turns out: $-\frac{32}{3}$



Not bad, but...

PCs easier!

$$\iint_R (2x-2y)\,dA = \int_{\theta=0}^{\theta=\pi}\int_{r=0}^{r=2}(2r\cos\theta-2r\sin\theta)r\,dr\,d\theta$$

$$= 2\int_{\theta=0}^{\theta=\pi}\int_{r=0}^{r=2} r(\cos\theta-\sin\theta)r\,dr\,d\theta$$

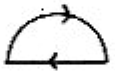
$$= 2\left[\int_0^2 r^2\,dr\right]\left[\int_0^{\pi}(\cos\theta-\sin\theta)\,d\theta\right]$$

$$= 2\left(\left[\frac{r^3}{3}\right]_0^2\right)\left(\left[\sin\theta+\cos\theta\right]_0^{\pi}\right)$$

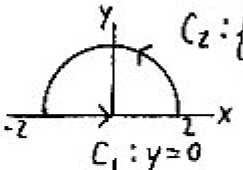
$$= 2\left(\frac{8}{3}\right)\left(\left[\underbrace{\sin\pi}_{=0}+\underbrace{\cos\pi}_{=-1}\right]-\left[\underbrace{\sin 0}_{=0}+\underbrace{\cos 0}_{1}\right]\right)$$

$$= \frac{16}{3}(-1-1)$$

$$= \boxed{-\frac{32}{3}}$$

Note 1: If  $\Rightarrow \int_{C\text{ in neg. direction}} y^2dx + x^2dy = \frac{32}{3}$

Note 2: 18.2 Method longer

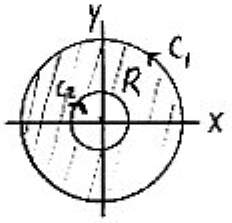


$C_2 : \begin{cases} x = 2\cos t \\ y = 2\sin t \end{cases}, \quad t: 0 \to \pi$

$C_1 : y = 0$

$x: -2 \to 2$

Ⓔ Extension

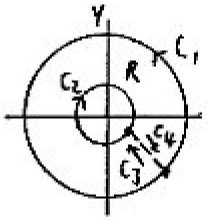


$$\oint_{C_1} \vec{F}\cdot d\vec{r} + \overset{R \text{ stays on left}}{\downarrow}\!\!\oint_{C_2} \vec{F}\cdot d\vec{r}$$

$$= \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA \quad \textcircled{\star}$$

Why?

Make a slit.

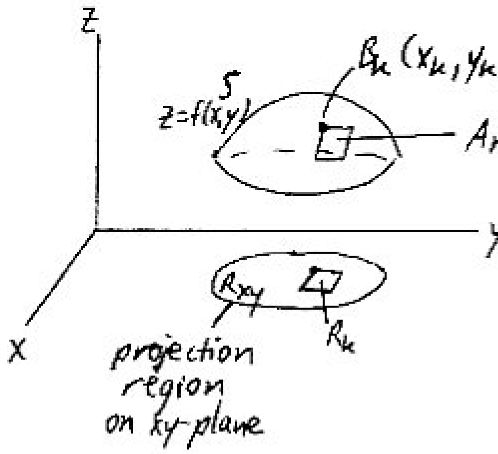


$$\textcircled{\star} = \oint_{C_1} \vec{F}\cdot d\vec{r} + \underbrace{\oint_{C_3} \text{`` ''}} + \oint_{C_2} \text{`` ''} + \underbrace{\oint_{C_4} \text{`` ''}} \text{ by Green.}$$

Sum $= 0$

18.5: SURFACE INTEGRALS

Ⓐ Review Surface Area (17.4)



$$B_k(x_k,y_k,z_k) = (x_k, y_k, \overbrace{f(x_k,y_k)}^{B_k \text{ is on } S})$$

Area of Sticky T_k (part of tangent plane at B_k over R_k)

$= \Delta T_k$

(Assume f is "nice" - cont. and has cont. 1st PDs on R_{xy}. If there is a problem at the boundary, we may need an improper integral.)

$$\text{Surface Area} = \iint_S dS$$
$$= \iint_{R_{xy}} \sqrt{1+[f_x(x,y)]^2+[f_y(x,y)]^2}\, dA$$

Similar if $y=f(x,z)$; $x=f(y,z)$.

Ⓑ Mass, m, of a Surface, S

$$\text{Mass of Sticky } T_k \approx \underbrace{\delta(x_k, y_k, f(x_k,y_k))}_{\substack{\text{Area mass density} \\ \text{at } B_k \ (\approx \text{same} \\ \text{throughout Sticky}) \\ (\text{Units like } g/m^2)}} \underbrace{\Delta T_k}_{\substack{\text{Area of} \\ \text{Sticky } T_k \\ (\text{Units like } m^2)}} \quad \leftarrow Ⓐ$$

$$m = \text{Mass of } S = \lim_{\|P\| \to 0} \sum_k Ⓐ$$
$$= \iint_S \delta(x,y,z)\, dS$$
$$= \iint_{R_{xy}} \delta(x,y,f(x,y)) \sqrt{1+[f_x(x,y)]^2+[f_y(x,y)]^2}\, dA$$

Ex (Like #1 - solutions manual flawed)

Find the mass of S, if $\delta(x,y,z) = x^2$, and S is the upper half of the sphere $x^2+y^2+z^2=a^2$, $(a>0)$.

Sol'n

Rewrite the eq. for S in the form $z=f(x,y)$.

$$x^2+y^2+z^2 = a^2 \qquad (a>0)$$
$$z = \pm\sqrt{a^2-x^2-y^2}$$

↑ we take the upper half of the sphere

Find m

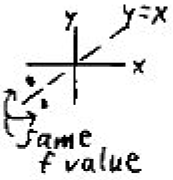
$$m = \iint_{R_{xy}} \delta \sqrt{1+[f_x(x,y)]^2+[f_y(x,y)]^2}\, dA$$

$$f(x,y) = \sqrt{a^2-x^2-y^2} = (a^2-x^2-y^2)^{1/2}$$

$$f_x(x,y) = \frac{1}{\cancel{2}}(a^2-x^2-y^2)^{-1/2}(-\cancel{2}x)$$
$$= -\frac{x}{\sqrt{a^2-x^2-y^2}}$$

$$[f_x(x,y)]^2 = \frac{x^2}{a^2-x^2-y^2}$$

$f(x,y)$ is symmetric in x and y (i.e., $f(x,y) = f(y,x)$).



$$\Rightarrow [f_y(x,y)]^2 = \frac{y^2}{a^2-y^2-x^2}$$

18.5.3

$$m = \iint_{R_{xy}} x^2 \sqrt{1 + \frac{x^2}{a^2-x^2-y^2} + \frac{y^2}{a^2-x^2-y^2}}\, dA$$

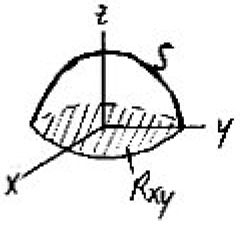
$$= \iint_{R_{xy}} x^2 \sqrt{1 + \frac{x^2+y^2}{a^2-x^2-y^2}}\, dA$$

1 is how many fourths?

$$= \iint_{R_{xy}} x^2 \sqrt{\frac{a^2 \cancel{-x^2} \cancel{-y^2} \cancel{+x^2} \cancel{+y^2}}{a^2-x^2-y^2}}\, dA$$

$$= \iint_{R_{xy}} x^2 \left(\frac{a}{\sqrt{a^2-x^2-y^2}}\right) dA$$

$\longrightarrow$ PCs



$$= \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=a} (r\cos\theta)^2 \left(\frac{a}{\sqrt{a^2-r^2}}\right) r\, dr\, d\theta$$

$$= \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=a} (r^2\cos^2\theta) \left(\frac{a}{\sqrt{a^2-r^2}}\right) r\, dr\, d\theta$$

$$= a \underbrace{\left[\int_0^{2\pi} \cos^2\theta\, d\theta\right]} \underbrace{\left[\int_0^a \frac{r^3}{\sqrt{a^2-r^2}}\, dr\right]}$$

$$\overset{PRI}{=} \int_0^{2\pi} \frac{1+\cos(2\theta)}{2} d\theta$$

Improper $\int$!!

$$= \frac{1}{2} \left[\theta + \frac{1}{2}\sin(2\theta)\right]_0^{2\pi}$$

(Integrand is undefined at $r=a$.)

$$= \frac{1}{2}\left(\left[2\pi + \cancelto{0}{\frac{1}{2}\sin(4\pi)}\right] - [0]\right)$$

$$= \boxed{\pi}$$

18.5.4

$$= \pi a \cdot \lim_{t \to a^-} \underbrace{\int_0^t \frac{r^3}{\sqrt{a^2-r^2}}\, dr}$$

Work out Indefinite $\int$, 1st.

Trig sub: $r = a \sin \gamma$ (We've had θ) or

Fancy u-sub

$$u = a^2 - r^2 \quad \Rightarrow \quad r^2 = a^2 - u$$
$$du = -2r\,dr$$
$$\Rightarrow r\,dr = -\tfrac{1}{2}\,du$$

$$\int \frac{r^3}{\sqrt{a^2-r^2}}\,dr = \int \frac{r^2 \cdot r}{\sqrt{a^2-r^2}}\,dr$$
$$= \int \frac{(a^2-u)(-\frac{1}{2}du)}{\sqrt{u}}$$
$$= -\frac{1}{2}\int (a^2 u^{-\frac{1}{2}} - u^{\frac{1}{2}})\,du$$
$$= -\frac{1}{2}\left[a^2\left(\frac{u^{1/2}}{1/2}\right) - \frac{u^{3/2}}{3/2}\right] + C$$
$$= -\frac{1}{2}\left[2a^2\sqrt{u} - \frac{2}{3}u^{3/2}\right] + C$$
$$= -a^2\sqrt{a^2-r^2} + \frac{1}{3}(a^2-r^2)^{3/2} + C$$

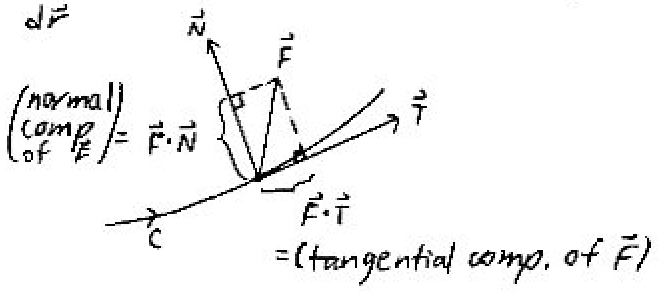
$$= \pi a \cdot \lim_{t \to a^-} \left[-a^2\sqrt{a^2-r^2} + \frac{1}{3}(a^2-r^2)^{3/2}\right]_0^t$$
$$= \pi a \cdot \lim_{t \to a^-} \left(\left[-a^2 \underbrace{\sqrt{a^2-t^2}}_{\to 0} + \frac{1}{3}\underbrace{(a^2-t^2)^{3/2}}_{\to 0}\right] - \left[-a^2 \underbrace{\sqrt{a^2}}_{=|a|=a} + \frac{1}{3}\underbrace{(a^2)^{3/2}}_{=a^3}\right]\right)$$
$$= \pi a\left(a^3 - \frac{1}{3}a^3\right)$$
$$= \pi a\left(\frac{2}{3}a^3\right)$$
$$= \boxed{\frac{2\pi a^4}{3}}$$

In upper division physics

Ⓒ Flux Integrals

(18.2: 2D/3D)

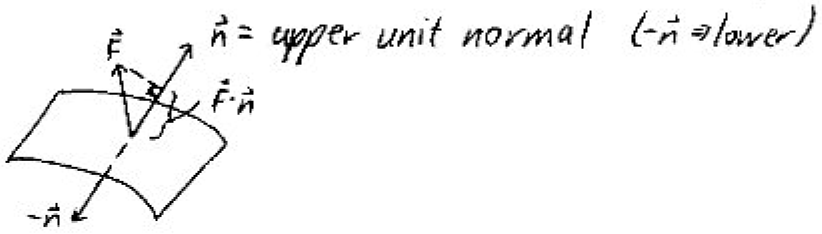
$$\int_C \vec F \cdot \underbrace{\vec T\, ds}_{d\vec r} = \text{Work done by } \vec F \text{ along } C.$$



(Now: 2D)

$$\int_C \vec F \cdot \vec N\, ds = \text{Flux (Latin for "flow") of } \vec F \text{ across } C.$$

(Now: 3D)



Work-Recall: $d\vec r = \vec T ds$

Flux-Now: $d\vec S = \vec n dS$

$$\iint_S \vec F \cdot \underbrace{\vec n\, dS}_{d\vec S} = \text{Flux of } \vec F \text{ across } S.$$

Sample units:

$$\underbrace{\underset{\text{"continuous addition"}}{\iint} \underbrace{\underbrace{\overset{\vec F}{\left\langle \frac{m}{sec}, \frac{m}{sec}, \frac{m}{sec} \right\rangle} \cdot \overset{\vec n}{\left(\text{no units}\right)}}_{\left(\frac{m}{sec}\right)} \overset{dS}{\left(m^2\right)}}_{\left(\frac{m^3}{sec}\right)}}_{\left(\frac{m^3}{sec}\right)}$$

$\frac{\text{Volume}}{\text{time}}$: flow rate across S $\left(\frac{m^3}{sec}\right)$

Let S be the graph of $z = f(x,y)$. What's $\vec{n}$?

$$\underbrace{z - f(x,y)}_{\text{"}g(x,y,z)\text{"}} = 0$$

$\Rightarrow$ S is a level surface of g.

$\Rightarrow$ $\vec{\nabla} g \perp S$

$\Rightarrow$ We use

$$\vec{n} = \frac{\vec{\nabla} g}{\|\vec{\nabla} g\|}$$

as our upper unit normal to S

$$= \frac{\left\langle \frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z} \right\rangle}{\|\vec{\nabla} g\|}$$

$$= \frac{\langle -f_x, -f_y, 1 \rangle}{\sqrt{1 + (f_x)^2 + (f_y)^2}}$$

$$\underbrace{z - f(x,y)}_{g(x,y,z)} = 0$$

Book does →

$$\iint_S \vec{F} \cdot \vec{n}\, dS = \iint_{R_{xy}} \vec{F} \cdot \frac{\overbrace{\langle -f_x, -f_y, 1 \rangle}^{\vec{\nabla} g}}{\cancel{\sqrt{1 + (f_x)^2 + (f_y)^2}}} \cancel{\sqrt{1 + (f_x)^2 + (f_y)^2}}\, dA$$

$$= \iint_{R_{xy}} \vec{F} \cdot \vec{\nabla} g\, dA$$

Ex Find the flux of $\vec{F}$ across S, where
$\vec{F}(x,y,z) = \langle z, 2, y \rangle$, and
S is the first-octant portion of the plane

$$z = -4x - 8y + 8.$$

Sol'n

$$\underbrace{z + 4x + 8y - 8}_{g(x,y,z)} = 0$$

$\vec{\nabla} g(x,y,z) = \langle 4, 8, 1 \rangle$ (This is a normal vector to this plane; see Ch.14 (14.5).)

$$\begin{aligned}
\text{Flux} &= \iint_{R_{xy}} \vec{F} \cdot \vec{\nabla} g \, dA \\
&= \iint_{R_{xy}} \langle z, 2, y \rangle \cdot \langle 4, 8, 1 \rangle \, dA \\
&= \iint_{R_{xy}} (4\enclose{circle}{z} + 16 + y) \, dA
\end{aligned}$$

write out in terms of x,y

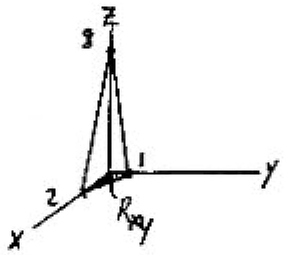
$$\begin{aligned}
&= \iint_{R_{xy}} [4(-4x - 8y + 8) + 16 + y] \, dA \\
&= \iint_{R_{xy}} [-16x - 31y + 48] \, dA
\end{aligned}$$

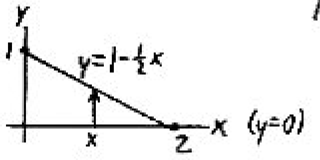
What is R_{xy}?

$$z = -4x - 8y + 8 \quad \text{in Octant I}$$

$$4x + 8y + z = 8$$

Intercept Method for Graphing a Plane





Intercept Form:

$$\frac{x}{2} + \frac{y}{1} = 1$$

$$y = 1 - \frac{1}{2}x$$

$$\text{Flux} = \int_{x=0}^{x=2} \int_{y=0}^{y=1-\frac{1}{2}x} [-16x - 31y + 48]\, dy\, dx$$

$$\vdots$$

$$= \boxed{27}$$

Units: maybe $\frac{m^3}{sec}$?

Note 1: We require that $\vec{n}$ be continuous over S, except on the boundary (i.e., S is orientable).

Note 2: We assume S has 2 sides (here: top, bottom). The Möbius strip is 1-sided and is not orientable.
see p.1003

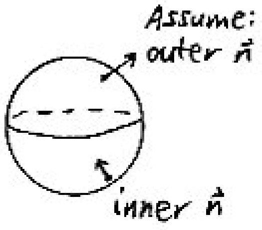
From Larson 6ed, 1049

We can parameterize such a surface:

$$\vec{r}(u,v) = \langle (4 - v\sin u)\cos(2u), (4 - v\sin u)\sin(2u), v\cos u \rangle$$

$$0 \le u \le \pi, \quad -1 \le v \le 1$$

Note 3: If S is closed, we have outer and inner normals.



$\Rightarrow$ Flux $= \iint_S \vec{F} \cdot \vec{n} \, dS$

= net outward flow across S

If $> 0 \Rightarrow$ There's a source of $\vec{F}$ within S.

If $< 0 \Rightarrow$ sink

If $= 0 \Rightarrow$ neither

Flux ( upper) + Flux ( lower)

A general advanced method can employ spherical coords. directly.

18.6: DIVERGENCE (or GAUSS's) THEOREM

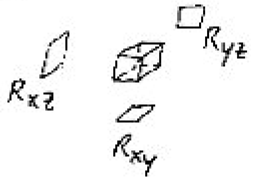
↳ discovered via electrostatics

also named after Carl Gauss (German mathem.!!, 1777-1855)
Michel Ostrogradsky (Russian mathem, 1801-61)

Larson 6ed - 1050

Often better than 18.5 for closed surfaces!

In 18.5,



Need 6 $\iint$s !!

Let S be a closed surface bounding a 3D region, Q.
Let $\vec n$ be the unit outer normal.
Let $\vec F$ have cont. PDs on Q.
Then,

$$\boxed{\underbrace{\iint_S \vec F\cdot\vec n\,dS}_{\text{Flux of }\vec F\text{ across }S} = \iiint_Q (\text{div}\,\vec F)\,dV}$$

Sample Units for Right-Hand Side
(18.5.5 for Left)

$$\underbrace{\iiint}_{\text{continuous addition}} \quad \underbrace{\text{div}\,\vec F}_{\frac{\partial M}{\partial x}+\frac{\partial N}{\partial y}+\frac{\partial P}{\partial z}} \quad dV$$

$$\left(\frac{m/sec}{m}+\dots\right) \quad (m^3)$$

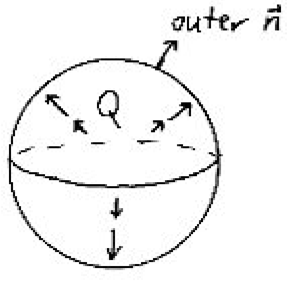
$$\left(\frac{m/sec}{m}\right) \quad (m^3)$$

$$\left(\frac{m^3}{sec}\right)$$

Note 1: (2D) Flux $= \oint_C \vec F\cdot\vec N\,ds \underset{\uparrow\ \text{by Green's Thm.}}{=} \iint_R (\text{div}\,\vec F)\,dA$

Note 2:

$\vec F$ vectors



Here, div $\vec F > 0$ throughout Q

⇒ Flux across S > 0

⇒ source in Q

Actually, the idea of $[\text{div}\,\vec F]_P$ comes from 18.6.

Ex (#8)

Find the flux of $\vec{F}(x,y,z) = \langle xy^2, yz^2, zx^2 \rangle$ through S, where S is the surface of the region between the cylinders $x^2+y^2=4$ and $x^2+y^2=9$ and between the planes $z=-1$ and $z=2$.

Sol'n

$$\text{Flux} = \iint_S \vec{F}\cdot\vec{n}\,dS$$

$$= \iiint_Q (\operatorname{div}\vec{F})\,dV$$

$$\operatorname{div}\vec{F} = \frac{\partial}{\partial x}(xy^2) + \frac{\partial}{\partial y}(yz^2) + \frac{\partial}{\partial z}(zx^2)$$

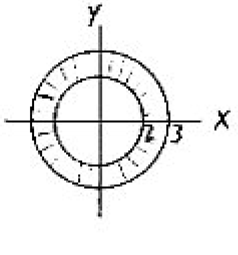
$$= y^2 + z^2 + x^2$$

$$= \underbrace{x^2+y^2}_{=r^2 \text{ in Cyl. Coords.}} + z^2$$

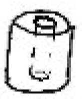
$$= r^2 + z^2$$

$$= \iiint_Q (r^2+z^2)\,r\,dr\,d\theta\,dz$$

R_{xy} :



Note S:



toilet paper roll

18.6.3

R_{xy}

$$= \int_{\theta=0}^{\theta=2\pi} \int_{r=2}^{r=3} \Big[\underbrace{\int_{z=-1}^{z=2} (r^2+z^2)r\, dz}\Big]\, dr\, d\theta$$

$$= \int_{z=-1}^{z=2} (r^3 + rz^2)\, dz$$

$$= \Big[r^3 z + r\Big(\frac{z^3}{3}\Big)\Big]_{z=-1}^{z=2}$$

$$= \Big(\Big[2r^3 + \frac{8}{3}r\Big] - \Big[-r^3 - \frac{1}{3}r\Big]\Big)$$

$$= (3r^3 + 3r)$$

$$= 3(r^3 + r)$$

$$= \Big[\int_0^{2\pi} d\theta\Big]\Big[\int_2^3 3(r^3+r)\, dr\Big]$$

$$= 2\pi\Big[3\Big(\frac{r^4}{4} + \frac{r^2}{2}\Big)\Big]_2^3$$

$$= 6\pi\Big(\Big[\frac{81}{4} + \frac{9}{2}\Big] - \underbrace{[4+2]}_{=6}\Big)$$

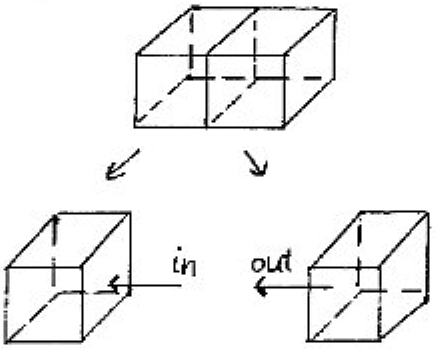
$$= 6\pi\Big(\frac{81 + 18 - 24}{4}\Big)$$

$$= \overset{3}{\cancel{6}}\pi\Big(\frac{75}{\underset{2}{\cancel{4}}}\Big)$$

$$= \boxed{\frac{225\pi}{2}} \text{ (flux units)}$$

Why Does the Theorem Work?

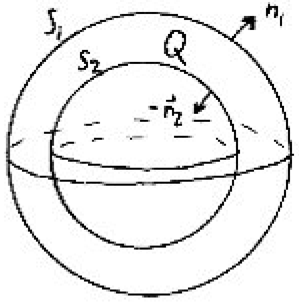
Roughly break Q into cubes.



Again, FTC idea here - boundary!! Same flavor as Green, Stokes, FTCI. Larson 6ed → 1054

The only net change of flow throughout Q will be at the boundary, S.
again, FTC idea here

Extension



$$\iiint_Q (\operatorname{div} \vec{F})\, dV$$

$$= \iint_{S_1} \vec{F} \cdot \vec{n}_1 \, dS + \iint_{S_2} \vec{F} \cdot (-\vec{n}_2)\, dS$$

Stewart 1137 5th ed 67 Used Thm. to prove Achimedes's principle

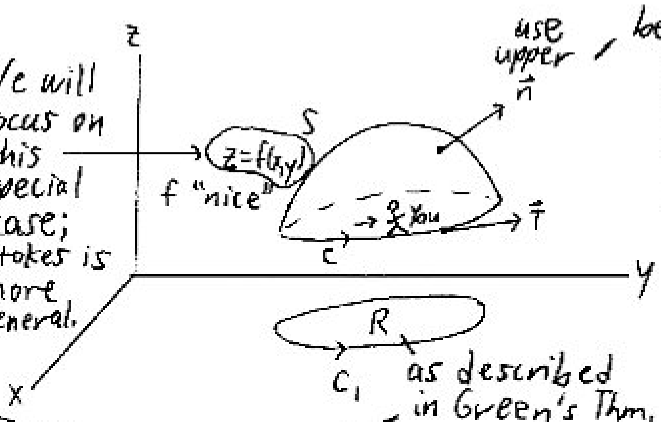
18.7: STOKES's THEOREM ("Green in 3D")

Stewart 1126 5ed ET

published in 1854 by George Stokes (Irish/Eng. mathem. physicist, 1819-1903). Lord Kelvin also key in both thms.; wrote letter to Stokes in 1850.

Broadly: Orientation, $\vec{n}$ consistent w/right-hand-rule idea.

We will focus on this special case; Stokes is more general.



use upper $\vec{n}$, because you must walk on top of C in the indicated direction in order to keep S to your left. This direction gives the positive orientation of C.

(vs.

lower $\vec{n}$)

R is "Green-type"

C_1: pcs simple closed curve
$R = C_1 \cup$ interior of C_1

as described in Green's Thm.

(See 18.5.6.)

$$\boxed{\text{Work} = \oint_C \vec{F}\cdot\vec{T}\,ds = \underbrace{\iint_S (\overrightarrow{\text{curl}}\,\vec{F})\cdot\vec{n}\,dS}_{}}\boxed{= \iint_{R_{xy}} (\overrightarrow{\text{curl}}\,\vec{F})\cdot\vec{\nabla}g\,dA}$$

Assume $\vec{F}$ is "nice" in an open region containing S.

= rotation of $\vec{F}$ about $\vec{n}$
= rot $\vec{F}$

where $g(x,y,z) = z - f(x,y)$

Other books: $\overrightarrow{\text{rot}}\,\vec{F} = \overrightarrow{\text{curl}}\,\vec{F}$

Swokowski →

= Flux of $\overrightarrow{\text{curl}}\,\vec{F}$ across S
= Surface $\iint$ of normal component of $\overrightarrow{\text{curl}}\,\vec{F}$ over S.

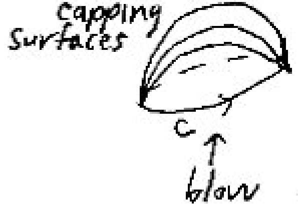
Green's and Stokes's Thms. are extensions of FTC. So is Div. Thm.

This is the 3D extension of Green's Thm., which can be expressed as: $\oint_C \vec{F}\cdot\vec{T}\,ds = \iint_R (\overrightarrow{\text{curl}}\,\vec{F})\cdot\textcircled{\vec{k}}\,dA$ in "vector form."

$\underbrace{\langle\ ,\ ,}_{\text{Given in }\mathbb{R}^2} 0\rangle$ ← Insert

Observe: If you think of "blowing a bubble (⊥ to xy-plane)" through C, you get the same value for the work integral!!

Term by Schey: 3ed, 92



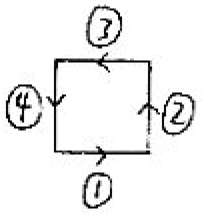
if $z = f(x,y)$ where f is "nice"; otherwise, more complicated story...

Stokes explains "paddlewheel" interpretation for $\overrightarrow{\text{curl}}$. (18.1.5)

Ex (#6)

If $\vec{F}(x,y,z) = \langle yz, xy, xz \rangle$, and C is the square with vertices $(0,0,2)$, $(1,0,2)$, $(1,1,2)$, and $(0,1,2)$, use Stokes' theorem to evaluate $\oint_C \vec{F} \cdot d\vec{r}$.

Without Stokes



$4 \int_S$

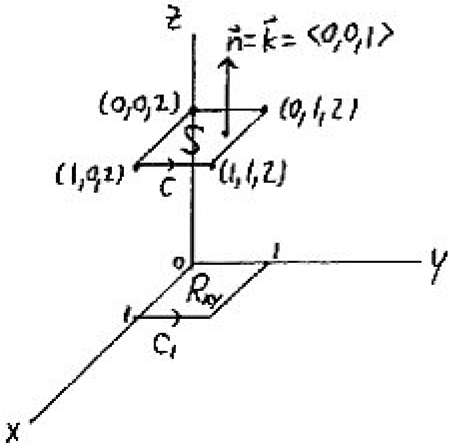
With Stokes

$$\overrightarrow{\text{curl}}\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & xy & xz \end{vmatrix}$$

$$= \left\langle \frac{\partial}{\partial y}(xz) - \frac{\partial}{\partial z}(xy), -\left[\frac{\partial}{\partial x}(xz) - \frac{\partial}{\partial z}(yz)\right], \frac{\partial}{\partial x}(xy) - \frac{\partial}{\partial y}(yz) \right\rangle$$

$$= \langle 0 - 0 \;, -[z \;-\; y] \;, \; y - z \rangle$$

$$= \langle 0, y-z, y-z \rangle$$



S lies on the graph of $z=2$.

$$\underbrace{z-2}_{g(x,y,z)}=0$$

$$\vec{\nabla} g(x,y,z)=\langle 0,0,1\rangle \quad (=\vec{k})$$

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_{R_{xy}} (\overrightarrow{\text{curl}}\,\vec{F})\cdot \vec{\nabla} g \, dA$$

$$= \iint_{R_{xy}} \langle 0, y-z, y-z\rangle \cdot \langle 0,0,1\rangle \, dA$$

$$= \iint_{R_{xy}} (y-\underset{\uparrow z=2}{z}) \, dA$$

<u>Note</u>: When $S=R_{xy}$, we end up with Green's Thm. Can you show this?

$$= \int_0^1 \int_0^1 (y-2)\, dx\, dy$$

$$= \underbrace{\left[\int_0^1 dx\right]}_{=1}\left[\int_0^1 (y-2)\, dy\right]$$

$$= \left[\frac{y^2}{2} - 2y\right]_0^1$$

$$= \left[\frac{(1)^2}{2} - 2(1)\right] - [0]$$

$$= \boxed{-\frac{3}{2}}$$

(Typed in review:)

When is $\vec{F}$ Conservative in $\mathbb{R}^3$? Equivalent statements:

In a connected region D (in which $\vec{F}$ is cont.)...
"in one piece"

Same as for $\mathbb{R}^2$ on 18.3.10, but

① $\vec{F}$ is conservative
(i.e., $\vec{F} = \vec{\nabla} f$ for some scalar potential func. f)

② We have IP: $\int_C \vec{F} \cdot d\vec{r}$

③ $\int_C \vec{F} \cdot d\vec{r} = 0$ for <u>every</u> simple closed curve C in D

replace (4a) → (4b) $\overrightarrow{\text{curl}}\, \vec{F} = \vec{0}$ throughout D
(i.e., $\vec{F}$ is <u>irrotational</u>)
if $\vec{F} = \langle M, N, P \rangle$ is "nice."

<u>Note</u>: $\int_C \vec{F} \cdot d\vec{r} = \int_C M\,dx + N\,dy + P\,dz$

If we start with (4b), then we require that D be <u>simply</u> connected.
different idea from $\mathbb{R}^2$ case

See pp. 1017-8.
This is s.c.:

A simple closed curve like this in D can be continuously deformed through D down to a pt. in D.

A donut (torus) is not.
There are simple closed curves in it that don't have capping surfaces.

Schey 3ed 100

<u>Note</u> (4b) is a very natural extension of (4a) into $\mathbb{R}^3$!!

$$\overrightarrow{\text{curl}}\ \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & P \end{vmatrix}$$

$$= \left\langle \frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}, -\left(\frac{\partial P}{\partial x} - \frac{\partial M}{\partial z}\right), \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right\rangle$$

$$= \left\langle \frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right\rangle$$

Observe:

$$\overrightarrow{\text{curl}}\ \vec{F} = \vec{0} \iff \frac{\partial P}{\partial y} = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = \frac{\partial P}{\partial x}, \text{and } \underbrace{\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}}_{\substack{\text{from (4a)} \\ \text{for } \mathbb{R}^2 !!}}$$

Also:

If $\vec{F}(x,y) = \langle M(x,y), N(x,y) \rangle$ is "nice" in $\mathbb{R}^2$, we can go to the $\mathbb{R}^3$ case by writing:

$$\vec{F}(x,y) = \langle M(x,y), N(x,y), \underset{\nwarrow \text{zero}}{0} \rangle$$

Then, $\overrightarrow{\text{curl}}\ \vec{F} = \langle 0, 0, \underbrace{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}_{\text{"scalar curl"}} \rangle$

This is $\vec{0} \Leftrightarrow \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$

Marsden 5ed, 556

SECTIONS 18.6 AND 18.7: ADDITIONAL NOTES AND REVISIONS

SECTION 18.6: DIVERGENCE (GAUSS'S) THEOREM

Instead of doing my Example in my notes (#8), I will do the following Example:

Example

Find the flux of $\mathbf{F}(x,y,z) = \langle 2x, x^2z^3, 5z \rangle$ through any sphere S of radius 4.

Solution

$$\text{Flux} = \iint_S \mathbf{F} \bullet \mathbf{n}\, dS = \iiint_Q (\text{div } \mathbf{F})\, dV \text{, where}$$

$$\begin{aligned}\text{div } \mathbf{F} &= \frac{\partial}{\partial x}(2x) + \frac{\partial}{\partial y}(x^2z^3) + \frac{\partial}{\partial z}(5z) \\ &= 2 + 0 + 5 \\ &= 7\end{aligned}$$

and Q is the region bounded by S.

$$\begin{aligned}\text{Flux} &= \iiint_Q 7\, dV \\ &= 7 \iiint_Q dV \\ &= 7(\text{Volume of } Q) \\ &= 7\left(\frac{4}{3}\pi(4)^3\right)\end{aligned}$$

since the volume of a sphere of radius r is $\frac{4}{3}\pi r^3$

$$\begin{aligned}&= 7\left(\frac{256\pi}{3}\right) \\ &= \frac{1792\pi}{3}\end{aligned}$$

SECTION 18.7: STOKES'S THEOREM

I will skip my Example (#6).

I may show in class why Green's Theorem is merely a special case of Stokes's Theorem.

We will make the usual assumptions for Stokes's Theorem. According to the theorem,

$$\text{Work } W = \oint_C \mathbf{F} \bullet \mathbf{T}\, ds = \iint_S \left(\textbf{curl } \mathbf{F}\right) \bullet \mathbf{n}\, dS$$

If S is a region of the xy-plane, we can call it R, and we use $\mathbf{n} = \mathbf{k}$:

$$\text{Work } W = \oint_C \mathbf{F} \bullet \mathbf{T}\, ds = \iint_R \left(\textbf{curl } \mathbf{F}\right) \bullet \mathbf{k}\, dA$$

This is called the vector form of Green's Theorem. Why?

Let $\mathbf{F}\left(x, y, 0\right) = \left\langle M\left(x, y\right), N\left(x, y\right), 0\right\rangle$.

You will find that $\left(\textbf{curl } \mathbf{F}\right) \bullet \mathbf{k} = \dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}$.

We then have:

$$\begin{aligned}\text{Work } W &= \oint_C \mathbf{F} \bullet \mathbf{T}\, ds = \iint_R \left(\textbf{curl } \mathbf{F}\right) \bullet \mathbf{k}\, dA \\ &= \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA\end{aligned}$$

The last expression should look familiar!

“COMING FULL CIRCLE” IN CALCULUS

The Generalized Stokes’s Theorem covers all of the major vector calculus theorems in this chapter, as well as the classic Fundamental Theorem of Calculus (FTC) from Calculus I.

What did the FTC say?

If f is integrable on the interval $\left[a,b\right]$ with antiderivative F on that interval,

$$\begin{aligned}\int_a^b f\left(x\right)\,dx &= \left[F\left(x\right)\right]_a^b \\ &= F\left(b\right)-F\left(a\right)\end{aligned}$$

The FTC relates an integral over an interval to information at the endpoints (the “boundary”) of that interval.

In Chapter 18, we related a higher-dimensional integral over a region to a lower-dimensional integral over the boundary of the region.

Calculus III is a very natural extension of Calculus I!!