CH. 18: VECTOR CALCULUS

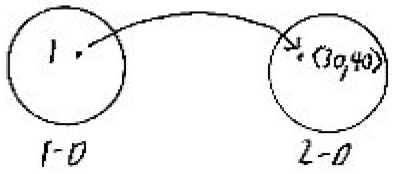
We'll look at extensions of the fund. Thm. of Calculus (FTC) from Calc I !!

18.1: VECTOR FIELDS

(A) Intro

Ch. 15

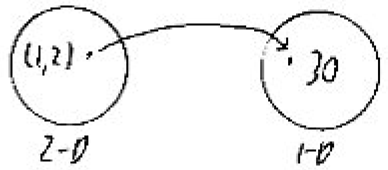
VVF $\vec{r}: D$, a subset of $\mathbb{R} \longrightarrow V_n$ (or $\mathbb{R}^n$)



$\vec{r}(1) = \langle 30, 40 \rangle$

Ch. 16

Scalar func. $f: D$, a subset of $\mathbb{R}^n \longrightarrow \mathbb{R}$

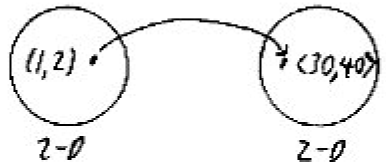


$f(1,2) = 30$

Now

A <u>vector field</u> is described by a

VVF $\vec{F}: D$, a subset of $\mathbb{R}^n \longrightarrow V_n$ (or $\mathbb{R}^n$)



$\vec{F}(1,2) = \langle 30, 40 \rangle$

We use P in $\vec{F} = \langle M, N, P \rangle$ in $\mathbb{R}^3$.

Every point A in a region D gets a vector $\vec{F}|_A$.

(x,y) $\qquad \vec{F}(x,y)$

(x,y,z) $\qquad \vec{F}(x,y,z)$

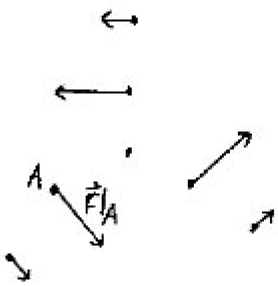
We'll study <u>steady vector fields</u>, in which vectors do not change with time.

<u>Ex</u> Velocity field for a kitchen sink (See 18.3.9, <u>Note 2</u>)

Show some of the vectors, enough to show a pattern.

Use **A** as the initial point for $\vec{F}|_A$.

Sample units: $\langle \frac{m}{sec}, \frac{m}{sec} \rangle$



<u>Ex</u> Electromagnetic force fields
Gravitational

Given a scalar multivariable func in Ch.16, what vector field did we construct? VVF

<u>Ex</u> (Ch. 16) Gradient vector field

<u>Ex</u> $f(x,y) = x^2 + y^2$ } Ch.16
$\vec{F}(x,y) = \vec{\nabla} f(x,y) = \langle 2x, 2y \rangle$ } Ch.18 (18.3)

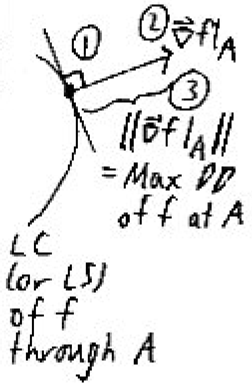
We call f a <u>potential function</u> for $\vec{F}$.
$-f$ in Physics so that: (Lower pot.) $\xleftarrow{\vec{F} = \vec{\nabla} f}$ (Higher pot.)

Like the GOP - must be conservative to have potential.

We call $\vec{F}$ a <u>conservative vector field</u>, because $\vec{F} = \vec{\nabla} f$ for some scalar func. f (i.e., $\vec{F}$ has a potential function).

Recall (from 16.6):

① $\underbrace{\vec{\nabla} f|_A}_{=\vec{F}|_A \text{ here}}$ $\perp$ level curve/surface (LC / LS) of the potential f containing A



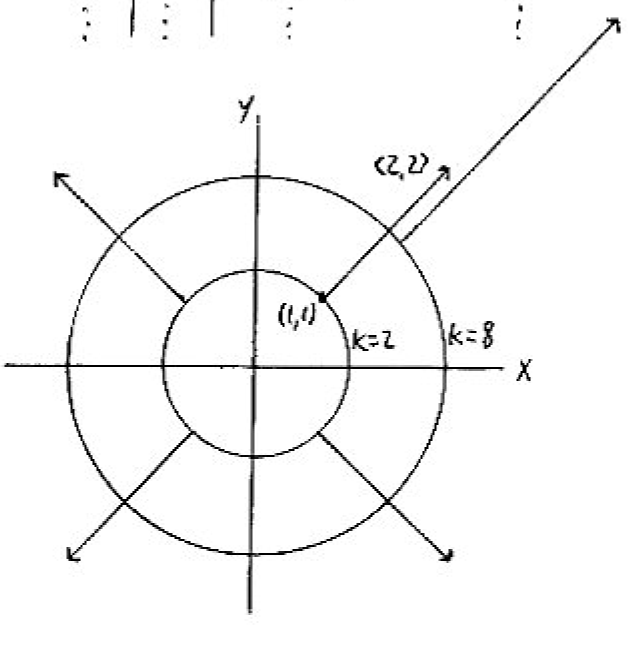
② $\vec{\nabla} f|_A$ points in direction of max rate of ↗ of f at A.

③ Its length, $\|\vec{\nabla} f|_A\|$ is that max rate of ↗.

Table

$\vec{F}|_A$
$\vec{\nabla} f|_A$

x (A)	y (A)	$\vec{F}(x,y) = \vec{\nabla} f(x,y) = \langle 2x, 2y\rangle$	(LC thru A) $k = f(x,y) = x^2+y^2$
1	1	$\langle 2,2\rangle$	2
-1	1	$\langle -2,2\rangle$	2
2	2	$\langle 4,4\rangle$	8
⋮	⋮	⋮	⋮



Ⓑ $\vec{\nabla}$ Operator
del/nabla

$$\vec{\nabla} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \quad \text{(Informal)}$$

$$\vec{\nabla} f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle \quad \text{gives a VVF (vector field)}$$

f ← scalar func.

} $\mathbb{R}^3$ versions

How does $\vec{\nabla}$ operate on VVFs?

(Assume $\vec{F}$ is "nice": components are cont. and have cont. 1^{st}-order PDs where we care.)

Marsden uses C^1 instead of "nice." "Smooth" implies derivs non-0, perhaps.

In 18.7, Swokowski says rot $\vec{F}$ = $(\overrightarrow{\text{curl}}\ \vec{F}) \cdot \vec{n}$

Ⓒ $\overrightarrow{\text{curl}}\ \vec{F} = \vec{\nabla} \times \vec{F}$ ($\vec{F}$ in $\mathbb{R}^3$)
(or rot)

(If $\vec{F} = \langle F_1, F_2 \rangle$ in $\mathbb{R}^2$, write $\langle F_1, F_2, 0 \rangle$) ← zero

gives a VVF (vector field)

Ex (#18) If $\vec{F}(x,y,z) = \langle \underbrace{x^3 \ln z}_{M(x,y,z)}, \underbrace{xe^{-y}}_{N(x,y,z)}, \underbrace{-(y^2+2z)}_{P(x,y,z)} \rangle$, find $\overrightarrow{\text{curl}}\ \vec{F}$.

$$\overrightarrow{\text{curl}}\ \vec{F} = \vec{\nabla} \times \vec{F}$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^3 \ln z & xe^{-y} & -(y^2+2z) \end{vmatrix}$$

$$= \left\langle \frac{\partial}{\partial y}[-(y^2+2z)] - \cancelto{0}{\frac{\partial}{\partial z}(xe^{-y})}, \right.$$

$$\ominus \left[\cancelto{0}{\frac{\partial}{\partial x}[-(y^2+2z)]} - \frac{\partial}{\partial z}(x^3 \ln z) \right],$$

$$\left. \frac{\partial}{\partial x}(xe^{-y}) - \cancelto{0}{\frac{\partial}{\partial y}(x^3 \ln z)} \right\rangle$$

$$= \left\langle -2y, x^3 \cdot \frac{1}{z}, e^{-y} \right\rangle$$

$$= \boxed{\left\langle -2y, \frac{x^3}{z}, e^{-y} \right\rangle}$$

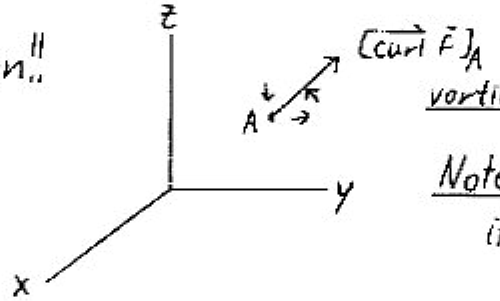
where $\vec{F}$ = velocity VVF

Don't need Cartesian!

Ⓓ Interpreting $[\overrightarrow{\text{curl}}\ \vec{F}]_A$ (in a coordinate-free sense)

Explained on pp. 1013-4 using Stokes's Thm. in 18.7.

"Local" rotation!!



vorticity vector

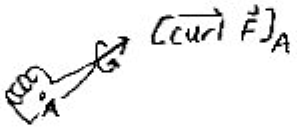
Note: $\vec{F}|_A$, itself, is irrelevant!! (Limit idea)

Direction of $[\overrightarrow{\text{curl}}\ \vec{F}]_A$ indicates the axis of rotation of $\vec{F}$ near A.

which way does $[\overrightarrow{\text{curl}}\ \vec{F}]_A$ point?

rotation is counter-clockwise if look from tip of $\overrightarrow{\text{curl}}$ vector towards the paddlewheel

Right-Hand Rule for $\overrightarrow{\text{curl}}$



curling of fingers indicate "overall rotation" (see ↓) of $\vec{F}$ near A

Idea

Analogous to idea of DDs at A max'ed in direction of $\vec{\nabla}f|_A$

At A, rotate this paddlewheel until its paddles revolve the fastest

See Swok 1016, MIT Munkres Calc II - F9

See HW

$\|[\overrightarrow{\text{curl}}\ \vec{F}]_A\|$ indicates the strength of the rotational effect about near A. It equals twice the angular speed of the paddles about this axis. Sample units: $\frac{\text{radians}}{\text{sec}}$ ← Time unit used for $\vec{F}$

$\overrightarrow{\text{curl}}\ \vec{F} = \vec{0}$ throughout $D \Leftrightarrow \vec{F}$ is irrotational in D

(E) $\operatorname{div} \vec{F} = \vec{\nabla} \cdot \vec{F}$

divergence gives a scalar function

Ex (#18) If $\vec{F}(x,y,z) = \langle x^3 \ln z,\ xe^{-y},\ -(y^2 + 2z) \rangle$, find $\operatorname{div} \vec{F}$.

$$\operatorname{div} \vec{F} = \vec{\nabla} \cdot \vec{F}$$

$$= \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle x^3 \ln z,\ xe^{-y},\ -(y^2+2z) \rangle$$

$$= \frac{\partial}{\partial x}(x^3 \ln z) + \frac{\partial}{\partial y}(xe^{-y}) + \frac{\partial}{\partial z}[-(y^2+2z)]$$

$$= 3x^2 \ln z + xe^{-y}(-1) + (-2)$$

$$= \boxed{3x^2 \ln z - xe^{-y} - 2}$$

Explained using Div. Thm. in 18.6

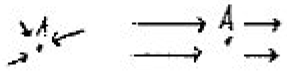
Stewart 1094 5ed ET

(F) Interpreting $[\operatorname{div} \vec{F}]_A$ (again, in terms of local behavior near A)

tendency of fluid to diverge from pt. A

If $[\operatorname{div} \vec{F}]_A < 0$, then there is a sink at A.

$\ominus$

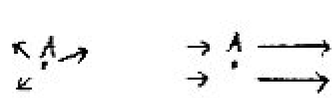
Tendency 

alphabetical order $\approx$ − to +

Ex cooling gas - compressible

If $[\operatorname{div} \vec{F}]_A > 0$, then there is a source at A.

$\oplus$



If $[\operatorname{div} \vec{F}]_A = 0 \Rightarrow$ neither. 

If $\operatorname{div} \vec{F} = 0$ throughout $D \Rightarrow \vec{F}$ is divergence free.

Exs incompressible fluids, solenoidal electromagnetic (EM) fields

incl. water, pretty much

See 18.6.1 for units.

(6) When is a Vector Field, $\vec{F}$, Conservative?

(Assume $\vec{F}$ is "nice.")

$\Leftrightarrow$ $\vec{F} = \vec{\nabla} f$ for some scalar func. f

$\Leftrightarrow$ $\overrightarrow{\text{curl}}\, \vec{F} = \vec{0}$ (if $\vec{F}$ nice in $\mathbb{R}^3$)

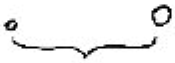
(You'll prove $\Rightarrow$ in HW! Converse ($\Leftarrow$) proven later in 18.7 on Stokes's Thm.)
#22

Marsden sed p.551: In $\mathbb{R}^3$, $\vec{F}$ can be conserv. even if undef. at a finite # of pts. (provided f also undef. there). Exceptional pts. not allowed in $\mathbb{R}^2$.

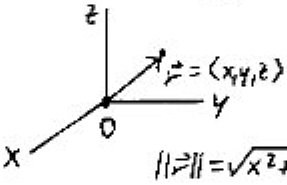
Larson: This is related to conservation of energy.
For a particle moving in a conservative force field,
the sum of its kinetic energy
due to motion
and its potential energy
due to position
is constant.

(H) Inverse Square Fields are Conservative

Ex Gravity, Electric force (Coulomb's Law)



The magnitude of the force between these is inversely proportional to the square of the distance between them. Let one be at O.



$2\times$ distance $\Rightarrow \frac{1}{4}$ of force
$3 \qquad\qquad \frac{1}{9}$

$\|\vec{r}\| = \sqrt{x^2+y^2+z^2}$

$$\vec{F}(x,y,z) = \underbrace{\left(\frac{c}{\|\vec{r}\|^2}\right)}_{\text{ensures that } \|\vec{F}(x,y,z)\| = \frac{|c|}{\|\vec{r}\|^2}}\underbrace{\left(\frac{\vec{r}}{\|\vec{r}\|}\right)}_{\text{unit vector in direction of } \vec{r}} \quad \text{for some constant of proportionality, } c.$$

$$= \frac{c\vec{r}}{\|\vec{r}\|^3}, \text{ provided } \vec{r} \neq \vec{0}$$

If $c>0$, repulsion (away from 0) If $c<0$, attraction (towards 0)

These vectors are $\perp$ to $\underbrace{\text{spheres centered at } O.}_{\text{level surfaces of potential}}$

In $\mathbb{R}^3$, so OK that $\vec{O}$ is an exceptional pt.

$\vec{F}$ is conserv.: $\vec{F} = \vec{\nabla} f$, where $f(x,y,z) = -\frac{c}{\|\vec{r}\|}$ (✓ this!!)

(I) Interesting IDs (in $\mathbb{R}^3$)

Stewart sec 17.5 p. 1091, 3: If $\vec{F}$ comps, f have cont. 2nd-order PDs

$\operatorname{div}(\overrightarrow{\operatorname{curl}}\,\vec{F}) = 0$

$\overrightarrow{\operatorname{curl}}\,(\underbrace{\vec{\nabla} f}_{\text{is conserv., remember?}}) = \vec{0}$ (see 18.1.3 figure)

div, $\overrightarrow{\text{curl}}$ critical to Maxwell's Laws in EM!!

Marsden 5ed, 290-2

(J) Flow Lines

(or Streamlines or Integral Curves)

If $\vec{F}$ is a velocity field, then a particle placed in the field will trace out a flow line.

(See 15.2.2)

Flow line carries "invisible" speed info.

If that's the path we want, how much does $\vec{F}$ help us?
What if we want another path?
Even if there's a flow line from pt. A to pt. B, can we do better?

We'll discuss Work in 18.2 (E).