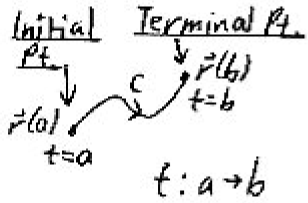
18.2: LINE (PATH) $\int$s

We'll do 2D, but this extends to 3D easily.

(A) Smooth Curves

$\vec{r}(t) = \langle x(t), y(t) \rangle$ gives a smooth parameterization of a curve, C, $\underbrace{\text{on } [a,b]}_{\text{i.e., when } a \le t \le b}$



$t: a \to b$

$\Leftrightarrow \vec{r}\,'(t) = \left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle$, a tangent VVF, is

① cont. on $[a,b]$, and
② never $\vec{0}$ on (a,b)

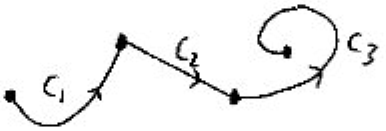
Then, C is a smooth curve with no breaks, corners, or cusps.

$\vec{r}$ can't backtrack.

(B) Piecewise-Smooth (ps) Curves

my notation

can be partitioned into a finite # of smooth curves.



$C = C_1 \cup C_2 \cup C_3$

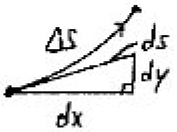
union

Ⓒ Mass, m, of a ps Wire, C

Recall (15.1)

ds = differential of arc length "s"

In Δt time,



$ds \approx \Delta s$

In 3D

$$\sqrt{\ldots + (dz)^2} \qquad = \sqrt{(dx)^2 + (dy)^2} \quad \text{(Informal)}$$

$$\sqrt{\ldots + \left(\tfrac{dz}{dt}\right)^2} \qquad = \sqrt{\left(\tfrac{dx}{dt}\right)^2 + \left(\tfrac{dy}{dt}\right)^2}\, dt \quad \left(\text{If } t\nearrow \text{; otherwise, "}|dt|\text{"}\right)$$

$$= \|\vec{r}\,'(t)\|\, dt \quad \text{or} \quad \underbrace{\underbrace{\|\vec{v}(t)\|}_{\text{speed}}\ \underbrace{dt}_{\Delta t = \text{change in time}}}_{\text{distance covered}}$$

Arc Length of C $= \int_C ds$

$$= \int_a^b \sqrt{\left(\tfrac{dx}{dt}\right)^2 + \left(\tfrac{dy}{dt}\right)^2}\, dt$$

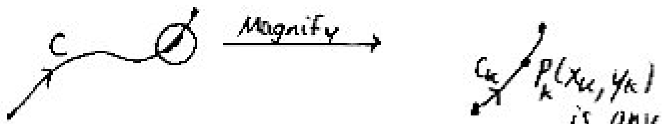
Now

Let $\delta(x,y)$ be the linear mass density of a wire at (x,y).

(Assume density is constant in a cross-section.)

Idea

Break C into tiny arcs.



$P_k(x_k, y_k)$ is any point on C_k.

$\delta \approx$ constant on C_k. ← tiny!

C_k: cover length Δs_k in time Δt_k

Mass of $C_k \approx$ (Density at P_k)(Arc length of C_k)

$= [\delta(x_k, y_k)][\Delta s_k]$

$$\boxed{\text{Mass, } m \text{, of } C = \int_C \delta(x,y)\, ds}$$

Why?

$$\int_C \delta(x,y)\,ds = \lim_{\underbrace{\|P\| \to 0}_{\text{largest } \Delta t_k}} \underbrace{\overbrace{\sum_k \underbrace{\delta(x_k, y_k)\,\Delta s_k}_{\text{Mass of } C_k}}^{\text{Riemann sum}}}_{\text{Approx. for } m}$$

This is an example of a...

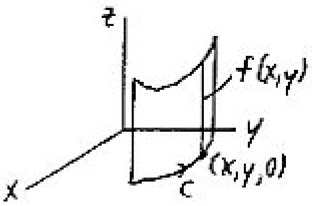
Stewart 5e, ET 1062: Invented in early 19c to study forces, fluid flow, EM

(D) Line (Path) Integral

$$\boxed{\int_C f(x,y)\,ds}$$

If $f(x,y)=1 \Rightarrow$ Arc Length of C
If $f(x,y)=\delta(x,y) \Rightarrow$ Mass of C

Lateral Surface Area ("Wall")



if $f(x,y)\geq 0$ along C

DETAILS...

<u>Note 1</u>: (We assume C is ps, and f is cont. on a region containing C.)

<u>Note 2</u>: We get the same value for $\int_C f(x,y)\,ds$, regardless of how we parameterize C. if smooth

Even the orientation doesn't matter:

$C \xrightarrow[\text{orientation}]{\text{reverse}} -C$

$$\int_C f(x,y)\,ds = \int_{-C} f(x,y)\,ds$$ ⇐ Expand using $|dt|$

If $\Delta s_k = 0$ $\Rightarrow \vec{r}\,'=0$ on C there, lose "smooth"

Always true: $\Delta s_k \geq 0$, mass ≥ 0
We flip sign if we had dx, dy, ...
(e.g., $\int_C f(t)\,dt = -\int_C f(t)\,dt$
$t: a \to b$ $t: b \to a$ for otherwise same param.)

Stewart 5e, ET 1067: → Remember $|dt|$ from ds

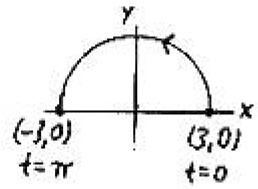
Like Ex 3

Ex Find the mass of a wire C if the density at $P(x,y)$ is directly proportional to its distance from the x-axis, and C is parameterized by $x=-3\cos t$, $y=3\sin t$; $0\le t\le \pi$.
(Assume $t: 0\to\pi$; $t\nearrow$ consistently w/orientation)

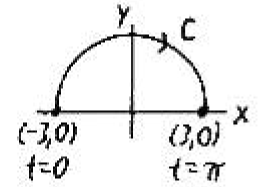
Sol'n

What is C? (Optional!)

If we had $\begin{cases} x = 3\cos t \\ y = 3\sin t \end{cases} \Rightarrow x^2+y^2=9$



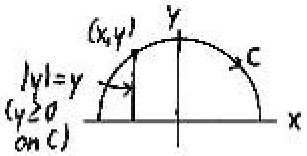
Here, $\begin{cases} x = \ominus 3\cos t \\ y = 3\sin t \end{cases}$



Good: No overlapping.

What is $\delta(x,y)$?

$\delta(x,y) = ky$



What is m?

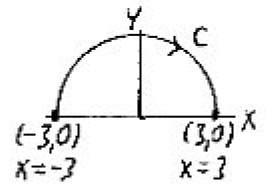
$$\begin{aligned} m &= \int_C \delta(x,y)\,ds \\ &= \int_0^{\pi} ky\sqrt{\left(\tfrac{dx}{dt}\right)^2+\left(\tfrac{dy}{dt}\right)^2}\,dt \\ &= \int_0^{\pi} k(3\sin t)\underbrace{\sqrt{(3\sin t)^2+(3\cos t)^2}}_{=\sqrt{9\sin^2 t+9\cos^2 t}\,=\,\sqrt{9(1)}\,=\,3}\,dt \end{aligned}$$

$x=-3\cos t$, $y=3\sin t$
$\frac{dx}{dt}=3\sin t$, $\frac{dy}{dt}=3\cos t$

$$\begin{aligned} &= 9k\left[-\cos t\right]_0^{\pi} \\ &= 9k\Big(\underbrace{-\underbrace{\cos\pi}_{(-1)}}_{1} - \underbrace{(-\underbrace{\cos 0}_{1})}_{(-1)}\Big) \\ &= \boxed{18k} \end{aligned}$$

If C lies on the graph of $y=f(x)$; $a \le x \le b \Rightarrow$
Let $x=t$, $y=f(t)$; $a \le t \le b$. (Similarly for $x=f(y)$.)

<u>Redo Previous Ex</u> (SKIPPED IN CLASS ?)



$$y=\sqrt{9-x^2}\ ,\ -3 \le x \le 3$$

$$x=t \quad ; \quad y=\sqrt{9-t^2} \text{ or } (9-t^2)^{1/2}$$

$$\frac{dx}{dt}=1 \quad ; \quad \frac{dy}{dt}=\frac{1}{2}(9-t^2)^{-1/2}(-2t) = -\frac{t}{\sqrt{9-t^2}}$$

$$-3 \le t \le 3 \quad (t: -3 \to 3)$$

$$m=\int_C \delta(x,y)\,ds = \underbrace{\int_{-3}^{3}}\ ky\sqrt{\left(\tfrac{dx}{dt}\right)^2+\left(\tfrac{dy}{dt}\right)^2}\,dt$$

(We basically could have used dx.)

or $2\int_0^3$ by sym. of C, $\delta(x,y)=ky$ even in x about $x=0$

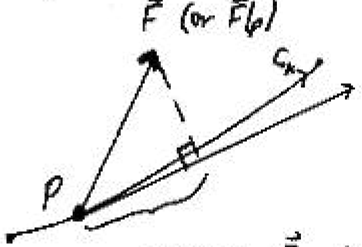
$$\begin{aligned}
&= 2\int_0^3 k\sqrt{9-t^2}\sqrt{(1)^2+\left(-\tfrac{t}{\sqrt{9-t^2}}\right)^2}\,dt\\
&= 2\int_0^3 k\sqrt{9-t^2}\sqrt{1+\tfrac{t^2}{9-t^2}}\,dt\\
&= 2\int_0^3 k\sqrt{(9-t^2)\left(1+\tfrac{t^2}{9-t^2}\right)}\,dt\\
&= 2\int_0^3 k\sqrt{9\cancel{-t^2+t^2}}\,dt\\
&= 2\int_0^3 3k\,dt\\
&= 6k\int_0^3 dt\\
&= 6k\,[t]_0^3\\
&= 6k(3)\\
&= \boxed{18k}
\end{aligned}$$

Ⓔ Line $\int$ of a Vector Field, $\vec{F}$

Let W = the work done by $\vec{F}$ on a particle moving along C [in the direction of orientation].

C_k, a tiny arc on C:

Normal component has no impact on the particle



$\text{comp}_{\vec{T}} \vec{F}$, the tangential component of $\vec{F}$

= ± magnitude of force acting in the direction of $\vec{T}$

$= \dfrac{\vec{F} \cdot \vec{T}}{\underbrace{\|\vec{T}\|}_{1}}$

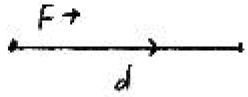
$= \vec{F} \cdot \vec{T}$

This depends on $\vec{F}$ and C, but not the particle's speed along C.

(We're measuring the impact that $\vec{F}$ has on the particle as it moves along C.)

Recall Calc II

If F is a constant scalar force, then $W = Fd$ here:



<u>Now</u> C curvy, $\vec{F}$ nonconstant

On tiny C_k, $\vec{F}\cdot\vec{T} \approx$ constant.

Work done along $C_k \approx \underbrace{(\vec{F}\cdot\vec{T})}_{\text{"force impact"}} \underbrace{\Delta s_k}_{\text{arc length of } C_k}$

$$\boxed{W = \int_C \underbrace{\vec{F}\cdot\vec{T}}_{\text{scalar function: special case of } f(x,y)}\, ds}$$

Stewart 5ed, ET, 1070

<u>Note</u> $\int_C \vec{F}\cdot\vec{T}\, ds = \ominus \int_{-C} \vec{F}\cdot\vec{T}\, ds$

actually $-(\text{old } \vec{T})$


$\vec{T}$


new $\vec{T}$

Orientation matters! Speed along C still doesn't.

$$= \int_C \vec{F}\cdot \underbrace{\frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|}}_{=\vec{T}(t)} \underbrace{\|\vec{r}\,'(t)\|\, dt}_{=ds}$$

$\leftarrow$ if $t\nearrow$, but formula still OK if $t \downarrow$ (!!) (see review notes)

$$\boxed{W = \int_C \vec{F}\cdot \underbrace{\vec{r}\,'(t)\, dt}_{=\frac{d\vec{r}}{dt}dt \;=\; \text{"}d\vec{r}\text{"}}}$$

Maybe best form if $\vec{F}, \vec{r}$ given in terms of t. ($\vec{F}$ only known for points along C. ~~???~~)

$$\boxed{W = \int_C \underbrace{\vec{F}}_{=\langle M(x,y),\, N(x,y)\rangle,\ \text{cont. in a region containing } C} \cdot \underbrace{d\vec{r}}_{\text{Think: } \langle dx, dy\rangle}}$$

$$\boxed{W = \int_C M\, dx + N\, dy}$$

() often omitted

<u>Differential Form</u>

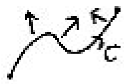
We'll use this a lot!

In 3D, $W = \int_C M\,dx + N\,dy + P\,dz$ if $\vec{F}(x,y,z) = \langle M(x,y,z), N(x,y,z), P(x,y,z)\rangle$

Idea

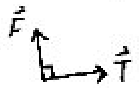
$W >> 0$ The force is with you!

$\vec{F}\cdot\vec{T} > 0$



$W = 0$ No help/harm.

$\vec{F}\cdot\vec{T} = 0$

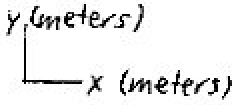


$W << 0$ Forces conspiring against you!

$\vec{F}\cdot\vec{T} < 0$

Swok 6.6
Also: ft.-lbs.

<u>Units</u> If $\vec{F}$ lengths in Newtons,



$\Rightarrow$ W in Newton-meters, or joules (J)

1 joule is the force needed to accelerate a 1 kg mass by $1\,\frac{m}{sec^2}$.

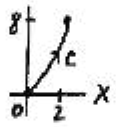
Like #4

<u>Ex</u> $\vec{F}(x,y) = \langle \underbrace{y}_{M(x,y)}, \underbrace{x+y}_{N(x,y)} \rangle$.

C is the graph of $y = x^2 + 2x$ directed from $(0,0)$ to $(2,8)$. Find the work, W, done by $\vec{F}$ on a particle moving along C.

<u>Sol'n</u>

Draw C (Optional?)



$C: y = \underbrace{x^2 + 2x}_{f(x)}$; $x: 0 \to 2$ (x is our parameter.)

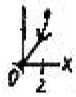
(Also nice: $x = f(y)$; $y: a \to b$)

Use Differential Form:

$$W = \int_C M\,dx + N\,dy$$
$$= \int_C y\,dx + (x+y)\,dy$$

$$\left.\begin{aligned} y &= x^2 + 2x \\ dy &= (2x+2)\,dx \end{aligned}\right\} \quad \left(\text{Write } \begin{aligned} x &= \ldots \\ dx &= \ldots \end{aligned} \ \text{ if } x = f(y)\right)$$

$$= \int_0^2 \underbrace{(x^2+2x)}_{y}\,dx + [x + \underbrace{(x^2+2x)}_{y}]\underbrace{(2x+2)\,dx}_{dy}$$
$$= \int_0^2 [x^2 + 2x + (x^2+3x)(2x+2)]\,dx$$
$$\vdots$$
$$= \boxed{48}$$

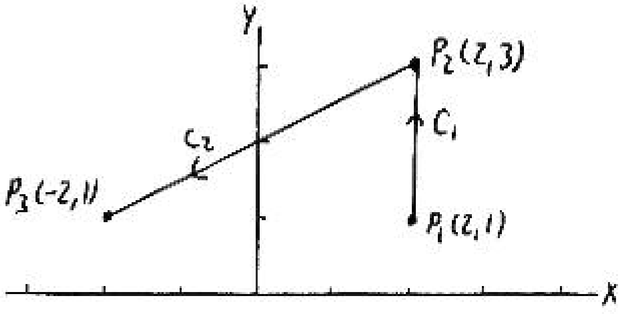
<u>Note</u>: If , then $x: 2 \to 0$, and $W = \int_2^0 \cdots = \boxed{-48}$. Makes sense!

OK if $x \downarrow$ in direction of motion.

$\vec{F}$ hurts us commensurately w/ how it helped us before.

<u>Ex</u> $\vec{F}(x,y) = \langle xy^2, e^{2y} \rangle$.

$C = C_1 \cup C_2$:



Find W.

<u>Method 1</u> (Know both methods!): Use t as a parameter.

ⓐ <u>Parameterize C_1, C_2</u>

C_1 $\quad C_1: \begin{cases} x = 2 \\ y = t \end{cases} \Rightarrow \begin{cases} dx = 0 \\ dy = dt \end{cases}$

$t: 1 \to 3$

C_2 $\quad$ Initial Point: $P_2(2,3)$

Displacement Vector: $\overrightarrow{P_2P_3} = \langle -2-2, 1-3 \rangle$
$= \langle -4, -2 \rangle$

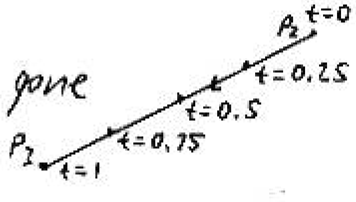
$C_2: \begin{cases} x = 2 - 4t \\ y = 3 - 2t \end{cases} \Rightarrow \begin{cases} dx = -4\,dt \\ dy = -2\,dt \end{cases}$

$t: 0 \to 1$

$\quad\ \ \Downarrow \qquad \Downarrow$

$\quad\ P_2 \qquad P_3$

t = fraction of the way you've gone from P_2 to P_3



(b) Find W

$$W = \int_{C_1} M\,dx + N\,dy \quad + \quad \int_{C_2} M\,dx + N\,dy$$

$$= \int_{C_1} xy^2\,dx + e^{2y}\,dy \quad + \quad \int_{C_2} xy^2\,dx + e^{2y}\,dy$$

$\leftarrow$ t-values $\leftarrow$

$$= \int_1^3 \underbrace{(2)(t)^2(0)}_{=0} + e^{2(t)}\,dt$$

$$+ \int_0^1 (2-4t)(3-2t)^2(-4)\,dt + e^{2(3-2t)}(-2)\,dt$$

$$\approx 198 - 209$$

$$= \boxed{-11}$$

Method 2: Use x and/or y as parameters.

(C_1) C_1: $x = 2 \Rightarrow dx = 0$
$y: 1 \to 3$

(C_2) Point: $(2,3)$
Slope $= \frac{1-3}{-2-2} = \frac{1}{2}$
Pt.-Slope Form: $y - 3 = \frac{1}{2}(x-2)$
$\Rightarrow$ Slope-Int. Form: $y = \frac{1}{2}x + 2$

C_2: $x: 2 \to -2$
$y = \frac{1}{2}x + 2 \Rightarrow dy = \frac{1}{2}dx$

$$W = \int_{C_1} M\,dx + N\,dy \quad + \quad \int_{C_2} M\,dx + N\,dy$$

$$= \int_{C_1} xy^2\,dx + e^{2y}\,dy \quad + \quad \int_{C_2} xy^2\,dx + e^{2y}\,dy$$

$\leftarrow$ y-values $\leftarrow$ $\qquad$ $\leftarrow$ x-values $\leftarrow$

$$= \int_1^3 \underbrace{(2)y^2(0)}_{=0} + e^{2y}\,dy \quad + \int_2^{-2} x\left(\tfrac{1}{2}x+2\right)^2 dx + e^{2(\frac{1}{2}x+2)} \cdot \tfrac{1}{2}\,dx$$

$\vdots$

$\approx \boxed{-11}$; same as for Method 1