18.3: INDEPENDENCE OF PATH (IP)

(A) Assumptions

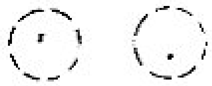
C is ps.

D is a simply connected open region containing C.

excludes boundary

"in one piece":
(any pair of points in D can be joined by a ps curve in D)

NO



Don't need until (E)

No Michigans

D has no holes (if in $\mathbb{R}^2$).
i.e., No simple closed curve in D encloses points not in D.

$\vec{r}(a) = \vec{r}(b)$, and the only self-intersection point is there.

In $\mathbb{R}^3$, can have finite # of exceptional pts. where $\vec{F}, f$ undef. for some of our Thms.

NO

Simply connected Ds in $\mathbb{R}^3$ discussed in 18.7 (pp. 1017-8). We'll extend (E) into 3D then, anyway.

$\vec{F}$ is cont. in D.

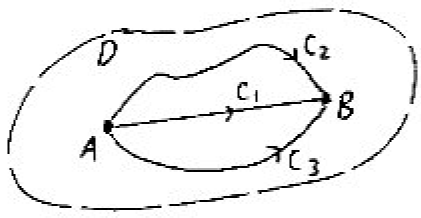
(B) Indep of Path (IP)

Find $\int_C \vec{F} \cdot d\vec{r}$.

What if C is hard to parameterize? $\sim_C$
Can we use $\rightarrow$, instead?

We have indep. of path (IP) for $\vec{F}$ in $D \Longleftrightarrow$

For **any** pair of points A and B in D, $\int_C \vec{F} \cdot d\vec{r}$ yields the same #, regardless of which ps curve C in D from A to B we use.



We can then say: $\int_A^B \vec{F} \cdot d\vec{r}$.

© Showing Indep. of Path $^{(IP)}$ for $\vec{F}$ in D by Finding a Potential Function, f

In D:

$\int_C \vec{F}\cdot d\vec{r}$ is indep. of path $_{(IP)}$ $\iff$ $\vec{F}$ is conservative $\iff$ $\vec{F} = \vec{\nabla} f$ for some scalar potential func., f

(Note that exists throughout D, though, in $\mathbb{R}^3$, we may have a finite # of exceptional pts. where $\vec{F}, f$ undefined. See 18.3.4 for an Ex in $\mathbb{R}^2$ where a suspected f doesn't work out.)

Proof pp. 982-4

Note: D need not be simply connected.
Need

Like #4

Ex $\vec{F}(x,y) = \langle 2xe^{2y} + 4y^3,\ 2x^2e^{2y} + 12xy^2 - 2y \rangle$.
Show that $\int_C \vec{F}\cdot d\vec{r}$ is indep. of path $_{(IP)}$ throughout $\mathbb{R}^2$.

Sol'n

Find a potential, f, such that

$$\vec{\nabla} f = \vec{F}$$

$$\langle f_x(x,y), f_y(x,y)\rangle = \langle 2xe^{2y} + 4y^3,\ \underbrace{2x^2e^{2y} + 12xy^2 - 2y}_{☆} \rangle$$

Partially $\int f_x$ wrt x $\quad f_x \nearrow f \quad$ ((Could $f \nwarrow f_y$ $\int$ wrt y)

$$\int f_x(x,y)\,dx = \int (2xe^{2y} + 4y^3)\,dx$$

$$f(x,y) = 2e^{2y}\left(\tfrac{x^2}{2}\right) + 4y^3x + \underbrace{g(y)}_{D_x[g(y)]=0}$$

$$f(x,y) = x^2e^{2y} + 4xy^3 + g(y) \quad ☆☆☆$$

<u>D_y both sides</u> $f \to f_y$

$$f_y(x,y) = D_y\left[x^2e^{2y} + 4xy^3 + g(y)\right]$$
$$= x^2(2e^{2y}) + 4x(3y^2) + g'(y)$$
$$= 2x^2e^{2y} + 12xy^2 + g'(y)$$

<u>Compare with ☆</u> $\Rightarrow f_y(x,y) = 2x^2e^{2y} + 12xy^2 - 2y$ ☆☆

<u>$\int$ ☆☆ wrt y to get g(y)</u>

$$\int g'(y)\,dy = \int -2y\,dy$$
$$g(y) = -y^2 + \underbrace{K}_{\text{"C" already used}}$$

<u>Write out f(x,y).</u> Use ☆☆☆.

In fact, this = $\int f_y(x,y)\,dy$

$$\boxed{f(x,y) = x^2e^{2y} + 4xy^3 - y^2 + K}$$ Can D_x, D_y to ✓.

Only Larson, as far as I know

<u>Alternative Method</u> (Don't use, though)

$$\int f_x(x,y)\,dy = \cdots = x^2e^{2y} + 4xy^3 + g(y)$$
$$\int f_y(x,y)\,dy = \cdots = x^2e^{2y} + 4xy^3 - y^2 + h(x)$$

Form a guess : $f(x,y) = x^2e^{2y} + 4xy^3 - y^2 + K$
D_x, D_y to ✓.

Ex Find a potential, f, for $\vec{F}(x,y,z) = \langle \underbrace{8x}_{f_x}, \underbrace{-9z}_{f_y}, \underbrace{-9y + 3z^2}_{f_z} \rangle$.

Sol'n

$f_x : 8x$

$$f : \int 8x\,dx = 4x^2 + \underbrace{g(y,z)}_{\text{Could have } y,\, z,\, \text{both, or neither}}$$

$f_y : g_y(y,z)$

$f_y : -9z$

$$\Downarrow$$

$$g_y(y,z) = -9z$$
$$g(y,z) = \int -9z\,dy$$
$$= -9zy + h(z)$$

$f : 4x^2 - 9yz + h(z)$ ✪

$f_z : -9y + h'(z)$

$f_z : -9y + 3z^2$

$$\Downarrow$$

$$-9y + h'(z) = -9y + 3z^2$$
$$h'(z) = 3z^2$$
$$h(z) = \int 3z^2\,dz$$
$$h(z) = z^3 + K$$

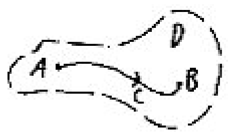
✪ $\Rightarrow$ $\boxed{f(x,y,z) = 4x^2 - 9yz + z^3 + K}$

Alternative Method (Don't use!)

$$\left.\begin{aligned} \int f_x(x,y,z)\,dx &= 4x^2 + g(y,z) \\ \int f_y(x,y,z)\,dy &= -9yz + h(x,z) \\ \int f_z(x,y,z)\,dz &= -9yz + z^3 + \ell(x,y) \end{aligned}\right\} \Rightarrow \text{Guess: } f(x,y,z) = 4x^2 - 9yz + z^3 + K$$

D_x, D_y, D_z to ✓.

Stewart calls **(D) Fundamental Thm. for Line $\int$s (FTLI)** - extends FTC from Calc I



If $\vec{F}$ is conservative in D with potential f $(\vec{F} = \vec{\nabla} f)$, then

$$\int_C \vec{F} \cdot d\vec{r} = \int_A^B \vec{F} \cdot d\vec{r} \quad \text{(by Indep. (IP) of Path for } \vec{F} \text{ in } D\text{)}$$

$$= \left[f(x,y) \right]_A^B \quad \text{or} \quad \left[f(x,y,z) \right]_A^B$$

$$= f\big|_B - f\big|_A$$

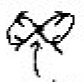
<u>Physical Interpretation</u>

The work done by a conservative force field $\vec{F}$ along any path C from A to B

= The difference in potentials between A and B.

Doesn't matter where you start. If do 1 full circuit ↺, same net effect. ↻: $-0=0$

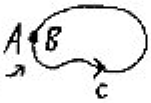
"Simple" helps. Orientation unclear for



Where do we go when we hit this pt.?

<u>What if $A=B$?</u>

closed curve



$$\int_C \vec{F} \cdot d\vec{r} = \int_A^A \vec{F} \cdot d\vec{r}$$

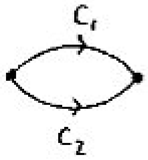
$$= f\big|_A - f\big|_A$$

$$= 0$$

True "Converse":

If $\int_C \vec{F}\cdot d\vec{r} = 0$ for every simple closed curve C in D, then $\vec{F}$ is conservative, and we have indep. of path for $\vec{F}$ in D.
(IP)

Why?



If $\int_C \vec{F}\cdot d\vec{r} = 0$

$\Leftrightarrow$

$\Rightarrow \int_{C_1} \text{`` ''} + \int_{-C_2} \text{`` ''} = 0$

$\Rightarrow \int_{C_1} \text{`` ''} - \int_{C_2} \text{`` ''} = 0$

$\Rightarrow \quad \int_{C_1} \text{`` ''} = \int_{C_2} \text{`` ''}$

Ex From ©: $\vec{F}(x,y) = \langle 2xe^{2y} + 4y^3,\ 2x^2e^{2y} + 12xy^2 - 2y \rangle$.
(*conservative in $\mathbb{R}^2$).

Evaluate $\int_{(1,2)}^{(3,4)} \vec{F}\cdot d\vec{r}$.

Method 1: Use FTLI.

$= \left[f(x,y) \right]_{(1,2)}^{(3,4)}$

$= \left[x^2e^{2y} + 4xy^3 - y^2 \right]_{(1,2)}^{(3,4)}$ from ©

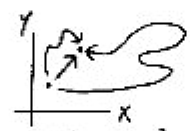
Don't need "+K."

752 + 9e⁸ → $752 + 9e^8$

$-[28 + e^4]$

$= \left[(3)^2e^{2(4)} + 4(3)(4)^3 - (4)^2 \right] - \left[(1)^2e^{2(2)} + 4(1)(2)^3 - (2)^2 \right]$

$= \boxed{9e^8 - e^4 + 724}$

whether we 

Method 2: Use Ⓔ, 18.2 method for "easy" C

Ⓔ Showing Indep. of Path (IP) for $\vec{F}$ in D (Method 2)

If $\overbrace{\vec{F} = \langle M, N \rangle}^{\text{in } \mathbb{R}^2}$, and $\overbrace{M, N \text{ have cont. } 1^{st} \text{ PDs}}^{\vec{F} \text{ is ``nice''}}$ Ⓐ on D, and if D is <u>simply</u> connected, then
no holes

#19: $\vec{F} = \langle M, N, P \rangle$: $M_y = N_x$, $M_z = P_x$, $N_z = P_y$

vs. $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$
My order better for 18.4

$$\boxed{\begin{array}{l} \int_C \vec{F} \cdot d\vec{r} = \int M\,dx + N\,dy \text{ is indep. of path (IP)} \\ \iff \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y} \quad \text{for } \underline{\text{all}} \ (x,y) \text{ in } D \end{array}}$$

Proof Idea

$(\Rightarrow)$ IP $\Rightarrow$ There exists f:

$$\begin{tikzcd}[column sep=small] & f \arrow[dl, "\partial_x"'] \arrow[dr, "\partial_y"] & \\ \vec{F}:\ M \arrow[dr, "\partial_y"'] & & N \arrow[dl, "\partial_x"] \\ & M_y = N_x & \end{tikzcd}$$

because $f_{xy} = f_{yx}$ if Ⓐ

$(\Leftarrow)$ Hard! Requires that D be <u>simply</u> connected.

<u>Ex</u> from Ⓒ: $\vec{F}(x,y) = \langle \underbrace{2xe^{2y} + 4y^3}_{M(x,y)}, \underbrace{2x^2e^{2y} + 12xy^2 - 2y}_{N(x,y)} \rangle$.
Assume 1^{st} PDs are cont. on $\mathbb{R}^2$.

Show that $\int_C \vec{F} \cdot d\vec{r}$ is indep. of path (IP) throughout $\mathbb{R}^2$.
(In Ⓒ, we did this by finding a potential.)

<u>Sol'n</u>

$$\frac{\partial N}{\partial x} = 4xe^{2y} + 12y^2$$

for all (x,y) in $\mathbb{R}^2$

$$\frac{\partial M}{\partial y} = 2x(2e^{2y}) + 12y^2 = 4xe^{2y} + 12y^2$$

" "

$$\Rightarrow \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$$

" "

$$\Rightarrow \int_C \vec{F} \cdot d\vec{r} \text{ is IP throughout } \mathbb{R}^2.$$

If C in D; here, $D = \mathbb{R}^2$.

<u>Note</u>: To compute $\int_{(1,2)}^{(3,4)} \vec{F} \cdot d\vec{r}$, we can use 18.2 on C: $(1,2) \to (3,4)$, $\left| \begin{array}{l} x = 1 + 2t \\ y = 2 + 2t \\ t: 0 \to 1 \end{array} \right|$

C can't pass over a "hole."

Note 1

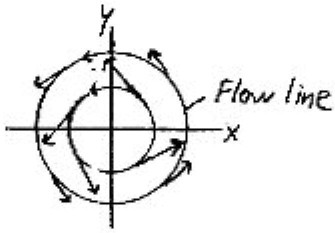
If $\frac{\partial N}{\partial x} \neq \frac{\partial M}{\partial y}$ at <u>any</u> (x,y) in D, then we do not have IP for $\vec{F}$ in D.

Like #27
$\langle -, + \rangle$

$\vec{F}$ is an approx. "kitchen sink" field. Vectors longer as $\|\vec{r}\|$ goes from $1 \to 0$
$\vec{F}$ undef.
This is "approximately" a sink; water doesn't move in!

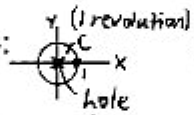
Note 2 (<u>How to Ace the Rest of Calculus</u>, pp. 234-5) (Skim in class)

If $\vec{F}(x,y) = \langle \frac{y}{x^2+y^2}, -\frac{x}{x^2+y^2} \rangle$, then



$\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$, <u>except</u> at $(0,0)$.

Turns out for:

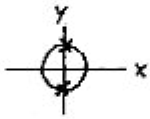


No <u>simply</u> connected region D contains C.

$\int_C \vec{F} \cdot d\vec{r} = 2\pi \neq 0$
↑ Using 18.2.

$\vec{F}$ is not conservative on any region containing C.

If you attempt to construct a potential, f, you may get $-\tan^{-1}\left(\frac{y}{x}\right)$, but observe that this is undefined when $x=0$.



Note 3

In 18.7, we will extend Ⓔ to 3D.

(Typed in review:)

(E) When is $\vec{F}$ Conservative in $\mathbb{R}^2$? Equivalent Statements:

In a connected region D (in which $\vec{F}$ is cont.)...
"in one piece"

(1) $\vec{F}$ is conservative
(i.e., $\vec{F} = \vec{\nabla} f$ for some scalar potential func. f)

(2) We have IP: $\int_C \vec{F} \cdot d\vec{r}$

(3) $\int_C \vec{F} \cdot d\vec{r} = 0$ for <u>every</u> simple closed curve C in D

(4a) $\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$ throughout D

if $\vec{F} = \langle M, N \rangle$ is "nice."

<u>Note</u>: $\int_C \vec{F} \cdot d\vec{r} = \int_C M\,dx + N\,dy$

If we start with (4a), then we require that D be <u>simply</u> connected.
no holes (in $\mathbb{R}^2$)

When we discuss the $\mathbb{R}^3$ case in 18.7, we will replace (4a) with (4b).