18.4: GREEN'S THEOREM

George Green (1793-1841) was an English mathematical physicist who published this theorem in an EM paper in 1828. Self-taught!

(A) Prelims

In $\mathbb{R}^2$:

Let C be a ps simple closed curve that is the boundary of $R = C \cup$ interior of C.

D is an open region containing R.

C is boundary of R: $C = \partial R$

in $\mathbb{R}^2$

Let $\vec{F} = \langle M, N \rangle$, where M, N are "nice" throughout D.

i.e., are cont. and have cont. 1^{st} PDs

$\oint_C \vec{F} \cdot d\vec{r}$

$\oint_C$ along C once in the positive direction

R always on the left

counterclockwise unless hole; See (C)

$\oint$ also used

Ⓑ Green's Thm

Green's Thm. an extension of FTC.

$$\boxed{\begin{aligned}\oint_C \vec{F}\cdot d\vec{r} &= \oint_C M\,dx + N\,dy \\ &= \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA\end{aligned}}$$

$= 0$ if $\vec{F}$ is conservative throughout D
(not "if and only if"; could be 0 even if $\vec{F}$ isn't)

Proof Note (FTC)
Fundamental Thm. of Calculus is used to equate a $\iint$ involving partials with a "1-D" $\int$ along a boundary.

Ⓒ Area of R

Note If you use $\vec{F}(x,y) = \langle \overset{M}{-y}, \overset{N}{x} \rangle$, then

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_R (1-(-1))\,dA$$
$$= 2\iint_R dA$$
$$= 2\cdot(\text{Area of } R)$$

$$\Rightarrow \boxed{\text{Area of } R = \frac{1}{2}\oint_C -y\,dx + x\,dy}$$

Often easier to use than: Area $= \oint_C x\,dy$ or $\oint_C -y\,dx$

How to Ace: You can judge a book by its cover!

Larson 5^e, ET 1023

Try this: Show that the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is πab. $(a>0, b>0)$

Hint: See my 13.1 Notes.

May be easier to remember:

$$\text{Area of } R = \frac{1}{2}\oint_C x\,dy - y\,dx$$

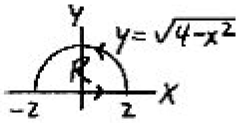
ⓓ Ex (#6)

Evaluate $\oint_C y^2\,dx + x^2\,dy$, where

C is the boundary of the region bounded by the semicircle $y=\sqrt{4-x^2}$ and the x-axis.

Sol'n

Draw C



$$\oint_C \underbrace{y^2}_{M}\,dx + \underbrace{x^2}_{N}\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA$$

(both nice)

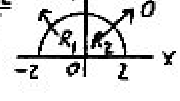
$$= \iint_R (2x - 2y)\, dA$$

In Cartesian coords.,

$$= \underbrace{\int_{x=-2}^{x=2}}_{} \int_{y=0}^{y=\sqrt{4-x^2}} (2x-2y)\, dy\, dx$$

NOT $\Rightarrow 2\int_0^2$, because not even in x.

Turns out: $-\frac{32}{3}$



Not bad, but...

PCs easier!

$$\iint_R (2x-2y)\,dA = \int_{\theta=0}^{\theta=\pi}\int_{r=0}^{r=2} (2r\cos\theta - 2r\sin\theta)\,r\,dr\,d\theta$$

$$= 2\int_{\theta=0}^{\theta=\pi}\int_{r=0}^{r=2} r(\cos\theta - \sin\theta)\,r\,dr\,d\theta$$

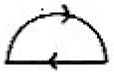
$$= 2\left[\int_0^2 r^2\,dr\right]\left[\int_0^{\pi}(\cos\theta - \sin\theta)\,d\theta\right]$$

$$= 2\left(\left[\frac{r^3}{3}\right]_0^2\right)\left(\left[\sin\theta + \cos\theta\right]_0^{\pi}\right)$$

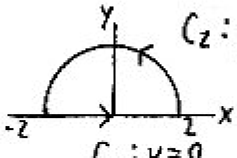
$$= 2\left(\frac{8}{3}\right)\left(\left[\underbrace{\sin\pi}_{=0} + \underbrace{\cos\pi}_{=-1}\right] - \left[\underbrace{\sin 0}_{=0} + \underbrace{\cos 0}_{1}\right]\right)$$

$$= \frac{16}{3}(-1-1)$$

$$= \boxed{-\frac{32}{3}}$$

Note 1: If  $\Rightarrow \int_{C \text{ in neg. direction}} y^2\,dx + x^2\,dy = \frac{32}{3}$

Note 2: 18.2 Method longer

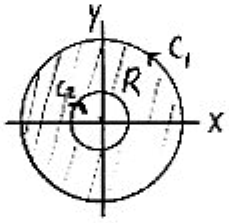


$C_2 : \begin{cases} x = 2\cos t \\ y = 2\sin t \end{cases}, \quad t: 0 \to \pi$

$C_1 : y = 0$

$x: -2 \to 2$

(E) Extension

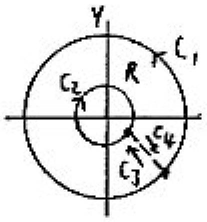


R stays on left

$$\oint_{C_1} \vec{F}\cdot d\vec{r} + \oint_{C_2} \vec{F}\cdot d\vec{r}$$
$$= \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA \quad (\star)$$

Why?

Make a slit.



$$(\star) = \oint_{C_1} \vec{F}\cdot d\vec{r} + \underbrace{\oint_{C_3} \text{''} + \oint_{C_2} \text{''} + \oint_{C_4} \text{''}}_{\text{Sum}=0} \text{ by Green.}$$