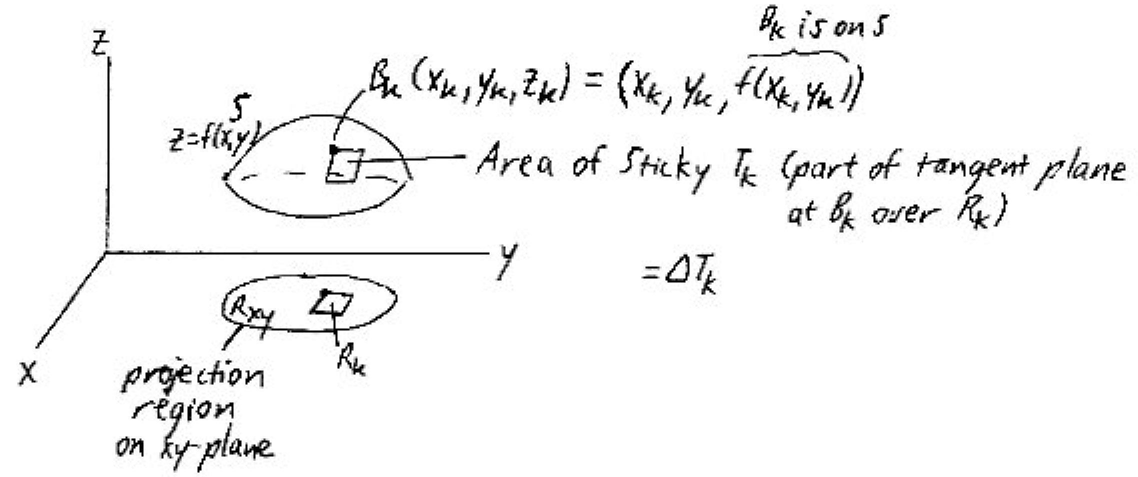
18.5: SURFACE INTEGRALS

Ⓐ Review Surface Area (17.4)



(Assume f is "nice" - cont. and has cont. 1^{st} PDs on R_{xy}. If there is a problem at the boundary, we may need an improper integral.)

$$\text{Surface Area} = \iint_S dS$$
$$= \iint_{R_{xy}} \sqrt{1 + [f_x(x,y)]^2 + [f_y(x,y)]^2}\, dA$$

Similar if $y = f(x,z)$; $x = f(y,z)$.

Ⓑ Mass, m, of a Surface, S

$$\text{Mass of Sticky } T_k \approx \underbrace{\delta(x_k, y_k, f(x_k, y_k))}_{\substack{\text{Area mass density at } P_k \text{ } (\approx \text{same throughout Sticky}) \\ (\text{Units like } g/m^2)}} \underbrace{\Delta T_k}_{\substack{\text{Area of Sticky } T_k \\ (\text{Units like } m^2)}} \quad \leftarrow ✪$$

$$m = \text{Mass of } S = \lim_{\|P\| \to 0} \sum_k ✪$$
$$= \iint_S \delta(x,y,z)\, dS$$
$$= \iint_{R_{xy}} \delta(x,y,f(x,y)) \sqrt{1 + [f_x(x,y)]^2 + [f_y(x,y)]^2}\, dA$$

<u>Ex</u> (Like #1 - solutions manual flawed)

Find the mass of S, if $\delta(x,y,z) = x^2$, and
S is the upper half of the sphere $x^2+y^2+z^2=a^2$, $(a>0)$.

<u>Sol'n</u>

<u>Rewrite the eq. for S in the form $z=f(x,y)$.</u>

$$x^2+y^2+z^2=a^2 \qquad (a>0)$$
$$z = \pm\sqrt{a^2-x^2-y^2}$$

↑we take the <u>upper</u> half of the sphere

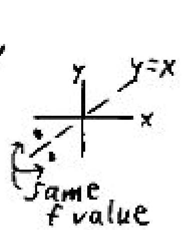
<u>Find m</u>

$$m = \iint_{R_{xy}} \delta \sqrt{1+[f_x(x,y)]^2+[f_y(x,y)]^2}\, dA$$

$$f(x,y) = \sqrt{a^2-x^2-y^2}$$
$$= (a^2-x^2-y^2)^{1/2}$$

$$f_x(x,y) = \tfrac{1}{2}(a^2-x^2-y^2)^{-1/2}(-2x)$$
$$= -\frac{x}{\sqrt{a^2-x^2-y^2}}$$
$$[f_x(x,y)]^2 = \frac{x^2}{a^2-x^2-y^2}$$

$f(x,y)$ is symmetric in x and y
(i.e., $f(x,y)=f(y,x)$).



$$\Rightarrow [f_y(x,y)]^2 = \frac{y^2}{a^2-y^2-x^2}$$

$$m = \iint_{R_{xy}} x^2 \sqrt{1 + \frac{x^2}{a^2-x^2-y^2} + \frac{y^2}{a^2-x^2-y^2}}\, dA$$

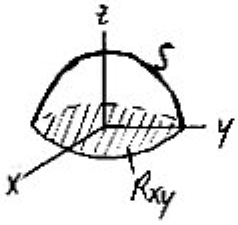
$$= \iint_{R_{xy}} x^2 \sqrt{1 + \frac{x^2+y^2}{a^2-x^2-y^2}}\, dA$$

1 is how many fourths?

$$= \iint_{R_{xy}} x^2 \sqrt{\frac{a^2 \cancel{-x^2} \cancel{-y^2} \cancel{+x^2} \cancel{+y^2}}{a^2-x^2-y^2}}\, dA$$

$$= \iint_{R_{xy}} x^2 \left(\frac{a}{\sqrt{a^2-x^2-y^2}}\right) dA$$

$\underline{\rightarrow PC_s}$



$$= \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=a} (r\cos\theta)^2 \left(\frac{a}{\sqrt{a^2-r^2}}\right) r\,dr\,d\theta$$

$$= \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=a} (r^2\cos^2\theta) \left(\frac{a}{\sqrt{a^2-r^2}}\right) r\,dr\,d\theta$$

$$= a \underbrace{\left[\int_0^{2\pi} \cos^2\theta\, d\theta\right]} \underbrace{\left[\int_0^a \frac{r^3}{\sqrt{a^2-r^2}}\, dr\right]}$$

$$\overset{PRI}{=} \int_0^{2\pi} \frac{1+\cos(2\theta)}{2}\, d\theta$$

Improper $\int$!!

$$= \frac{1}{2}\left[\theta + \frac{1}{2}\sin(2\theta)\right]_0^{2\pi}$$

(Integrand is undefined at $r=a$.)

$$= \frac{1}{2}\left(\left[2\pi + \cancel{\tfrac{1}{2}\sin(4\pi)}^{\,0}\right] - [0]\right)$$

$$= \pi$$

$$= \pi a \cdot \lim_{t \to a^-} \underbrace{\int_0^t \frac{r^3}{\sqrt{a^2-r^2}}\,dr}$$

<u>Work out Indefinite $\int$, 1st.</u>

Trig sub: $r = a\sin\gamma$ (We've had θ) <u>or</u>

Fancy u-sub

$$u = a^2 - r^2 \quad \Rightarrow \quad r^2 = a^2 - u$$
$$du = -2r\,dr$$
$$\Rightarrow r\,dr = -\tfrac{1}{2}\,du$$

$$\int \frac{r^3}{\sqrt{a^2-r^2}}\,dr = \int \frac{r^2 \cdot r}{\sqrt{a^2-r^2}}\,dr$$
$$= \int \frac{(a^2-u)(-\frac{1}{2}du)}{\sqrt{u}}$$
$$= -\frac{1}{2}\int (a^2u^{-\frac{1}{2}} - u^{\frac{1}{2}})\,du$$
$$= -\frac{1}{2}\left[a^2\left(\frac{u^{1/2}}{1/2}\right) - \frac{u^{3/2}}{3/2}\right] + C$$
$$= -\frac{1}{2}\left[2a^2\sqrt{u} - \frac{2}{3}u^{3/2}\right] + C$$
$$= -a^2\sqrt{a^2-r^2} + \frac{1}{3}(a^2-r^2)^{3/2} + C$$

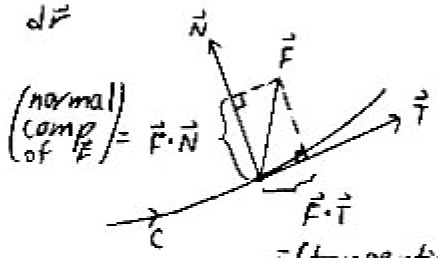
$$= \pi a \cdot \lim_{t\to a^-}\left[-a^2\sqrt{a^2-r^2} + \frac{1}{3}(a^2-r^2)^{3/2}\right]_0^t$$
$$= \pi a \cdot \lim_{t\to a^-}\left(\left[-a^2\underbrace{\sqrt{a^2-t^2}}_{\to 0} + \frac{1}{3}\underbrace{(a^2-t^2)^{3/2}}_{\to 0}\right] - \left[-a^2\underbrace{\sqrt{a^2}}_{=|a|=a} + \frac{1}{3}\underbrace{(a^2)^{3/2}}_{=a^3}\right]\right)$$
$$= \pi a\left(a^3 - \frac{1}{3}a^3\right)$$
$$= \pi a\left(\frac{2}{3}a^3\right)$$
$$= \boxed{\frac{2\pi a^4}{3}}$$

In upper division physics

Ⓒ Flux Integrals

(18.2: 2D/3D)

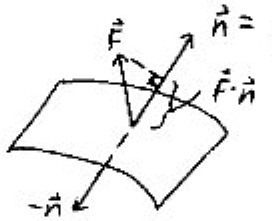
$$\int_C \vec{F}\cdot\underbrace{\vec{T}\,ds}_{d\vec{r}} = \text{Work done by } \vec{F} \text{ along } C.$$



(Now: 2D)

$$\int_C \vec{F}\cdot\vec{N}\,ds = \text{Flux (Latin for "flow") of } \vec{F} \text{ across } C.$$

(Now: 3D)



Work-Recall: $d\vec{r} = \vec{T}ds$
Flux-Now: $d\vec{S} = \vec{n}dS$

$$\iint_S \vec{F}\cdot\underbrace{\vec{n}\,dS}_{d\vec{S}} = \text{Flux of } \vec{F} \text{ across } S.$$

Sample units: $\iint$ "continuous addition" $\underbrace{\underbrace{\underbrace{\underset{\langle \frac{m}{sec}, \frac{m}{sec}, \frac{m}{sec}\rangle}{\vec{F}} \cdot \underset{(\text{no units})}{\vec{n}}}_{\left(\frac{m}{sec}\right)}\ \underset{(m^2)}{dS}}_{\left(\frac{m^3}{sec}\right)}}_{\left(\frac{m^3}{sec}\right)}$

$\frac{\text{Volume}}{\text{time}}$: flow rate across S $\left(\frac{m^3}{sec}\right)$

Let S be the graph of $z = f(x,y)$. What's $\vec{n}$?

$$\underbrace{z - f(x,y)}_{\text{"}g(x,y,z)\text{"}} = 0$$

$\Rightarrow$ S is a level surface of g.

$\Rightarrow$ $\vec{\nabla} g \perp S$

$\Rightarrow$ We use

$$\vec{n} = \frac{\vec{\nabla} g}{\|\vec{\nabla} g\|}$$

as our upper unit normal to S

$$= \frac{\left\langle \frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z} \right\rangle}{\|\vec{\nabla} g\|}$$

$$= \frac{\langle -f_x, -f_y, 1 \rangle}{\sqrt{1 + (f_x)^2 + (f_y)^2}}$$

$$\underbrace{z - f(x,y)}_{g(x,y,z)} = 0$$

Book does $\rightarrow$

$$\iint_S \vec{F} \cdot \vec{n}\, dS = \iint_{R_{xy}} \vec{F} \cdot \frac{\overbrace{\langle -f_x, -f_y, 1 \rangle}^{\vec{\nabla} g}}{\cancel{\sqrt{1 + (f_x)^2 + (f_y)^2}}} \cancel{\sqrt{1 + (f_x)^2 + (f_y)^2}}\, dA$$

$$= \iint_{R_{xy}} \vec{F} \cdot \vec{\nabla} g\, dA$$

Ex Find the flux of $\vec{F}$ across S, where
$\vec{F}(x,y,z) = \langle z, 2, y \rangle$, and
S is the first-octant portion of the plane

$$z = -4x - 8y + 8.$$

Sol'n

$$\underbrace{z + 4x + 8y - 8}_{g(x,y,z)} = 0$$

$\vec{\nabla} g(x,y,z) = \langle 4, 8, 1 \rangle$ (This is a normal vector to this plane; see Ch.14 (14.5).)

$$\text{Flux} = \iint_{R_{xy}} \vec{F} \cdot \vec{\nabla} g \, dA$$

$$= \iint_{R_{xy}} \langle z, 2, y \rangle \cdot \langle 4, 8, 1 \rangle \, dA$$

$$= \iint_{R_{xy}} (4z + 16 + y) \, dA$$

write out in terms of x,y

$$= \iint_{R_{xy}} [4(-4x - 8y + 8) + 16 + y] \, dA$$

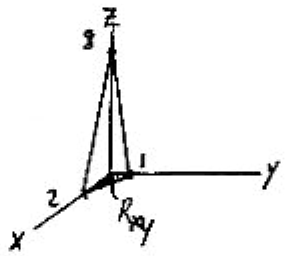
$$= \iint_{R_{xy}} [-16x - 31y + 48] \, dA$$

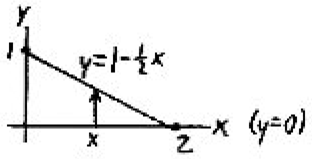
What is R_{xy}?

$$z = -4x - 8y + 8 \quad \text{in Octant I}$$

$$4x + 8y + z = 8$$

Intercept Method for Graphing a Plane





Intercept Form:

$$\frac{x}{2} + \frac{y}{1} = 1$$

$$y = 1 - \tfrac{1}{2}x$$

$$\text{Flux} = \int_{x=0}^{x=2} \int_{y=0}^{y=1-\frac{1}{2}x} [-16x - 31y + 48] \, dy \, dx$$

$$\vdots$$

$$= \boxed{27}$$

Units: maybe $\frac{m^3}{sec}$?

<u>Note 1</u>: We require that $\vec{n}$ be continuous over S, except on the boundary (i.e., S is <u>orientable</u>).

<u>Note 2</u>: We assume S has 2 sides (here: top, bottom). The <u>Möbius strip</u> is 1-sided and is not orientable.

see p.1003

From Larson 6ed, 1049

We can parameterize such a surface:

$$\vec{r}(u,v) = \langle (4 - v\sin u)\cos(2u),\ (4 - v\sin u)\sin(2u),\ v\cos u \rangle$$

$$0 \le u \le \pi, \quad -1 \le v \le 1$$

<u>Note 3</u>: If S is closed, we have <u>outer</u> and <u>inner</u> normals.

Assume: outer $\vec{n}$ $\Rightarrow$ Flux $= \iint_S \vec{F} \cdot \vec{n}\, dS$

= net <u>outward</u> flow across S

If $> 0 \Rightarrow$ There's a <u>source</u> of $\vec{F}$ within S.

If $< 0 \Rightarrow$ <u>sink</u>

If $= 0 \Rightarrow$ neither

Flux ( upper) + Flux ( lower)

A general advanced method can employ spherical coords. directly.