Larson 6ed - 1050

# 18.6: DIVERGENCE THEOREM (or GAUSS's)

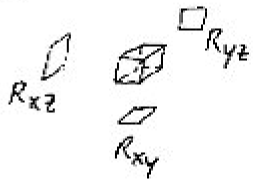
discovered via electrostatics

also named after Carl Gauss (German mathem. !!, 1777-1855)

Michel Ostrogradsky (Russian mathem, 1801-61)

Often better than 18.5 for closed surfaces!

In 18.5,




Need 6 $\iint$s !!

Let $S$ be a closed surface bounding a 3D region, $Q$.
Let $\vec{n}$ be the unit outer normal.
Let $\vec{F}$ have cont. PDs on $Q$.
Then,

$$\boxed{\underbrace{\iint_S \vec{F}\cdot\vec{n}\, dS}_{\text{Flux of } \vec{F} \text{ across } S} = \iiint_Q (\operatorname{div} \vec{F})\, dV}$$

Sample Units for Right-Hand Side (18.5.5 for Left)

$$\underbrace{\iiint}_{\substack{\text{continuous}\\ \text{addition}}} \quad \underbrace{\operatorname{div} \vec{F}}_{\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}} \quad dV$$

$\left(\frac{m/sec}{m} + \dots\right)$ $(m^3)$

$\left(\frac{m/sec}{m}\right)$ $(m^3)$

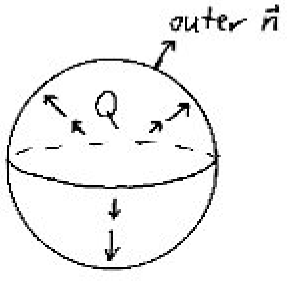
$\left(\frac{m^3}{sec}\right)$

**Note 1:** (2D) $Flux = \oint_C \vec{F}\cdot\vec{N}\, ds \underset{\substack{\uparrow\\ \text{by Green's Thm.}}}{=} \iint_R (\operatorname{div} \vec{F})\, dA$


**Note 2:**

Actually, the idea of $[\operatorname{div} \vec{F}]_P$ comes from 18.6.

$\vec{F}$ vectors




Here, $\operatorname{div} \vec{F} > 0$ throughout $Q$

$\Rightarrow$ Flux across $S > 0$

$\Rightarrow$ source in $Q$

Ex (#8)

Find the flux of $\vec{F}(x,y,z) = \langle xy^2, yz^2, zx^2 \rangle$ through $S$, where $S$ is the surface of the region between the cylinders $x^2+y^2=4$ and $x^2+y^2=9$ and between the planes $z=-1$ and $z=2$.

Sol'n

$$\text{Flux} = \iint_S \vec{F} \cdot \vec{n}\, dS$$

$$= \iiint_Q (\text{div}\, \vec{F})\, dV$$

$$\text{div}\, \vec{F} = \frac{\partial}{\partial x}(xy^2) + \frac{\partial}{\partial y}(yz^2) + \frac{\partial}{\partial z}(zx^2)$$

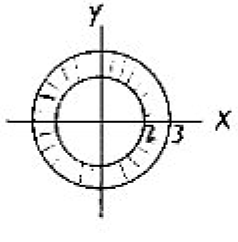
$$= y^2 + z^2 + x^2$$

$$= \underbrace{x^2 + y^2}_{= r^2 \text{ in Cyl. Coords.}} + z^2$$

$$= r^2 + z^2$$

$$= \iiint_Q (r^2 + z^2)\, r\, dr\, d\theta\, dz$$

$R_{xy}$ :




Note

$S$:

toilet paper roll

$$= \int_{\theta=0}^{\theta=2\pi} \int_{r=2}^{r=3} \underbrace{\left[\int_{z=-1}^{z=2} (r^2+z^2)r\, dz\right]}_{} dr\, d\theta$$

$$= \int_{z=-1}^{z=2} (r^3+rz^2)\, dz$$

$$= \left[r^3 z + r\left(\frac{z^3}{3}\right)\right]_{z=-1}^{z=2}$$

$$= \left(\left[2r^3 + \tfrac{8}{3}r\right] - \left[-r^3 - \tfrac{1}{3}r\right]\right)$$

$$= (3r^3 + 3r)$$

$$= 3(r^3+r)$$

$$= \left[\int_0^{2\pi} d\theta\right]\left[\int_2^3 3(r^3+r)\, dr\right]$$

$$= 2\pi\left[3\left(\frac{r^4}{4} + \frac{r^2}{2}\right)\right]_2^3$$

$$= 6\pi\left(\left[\tfrac{81}{4} + \tfrac{9}{2}\right] - \underbrace{[4+2]}_{=6}\right)$$

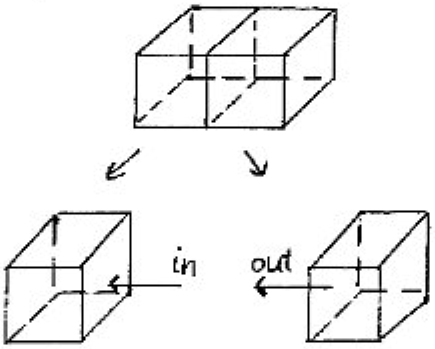
$$= 6\pi\left(\frac{81+18-24}{4}\right)$$

$$= \overset{3}{\cancel{6}}\pi\left(\frac{75}{\underset{2}{\cancel{4}}}\right)$$

$$= \boxed{\frac{225\pi}{2}} \text{ (flux units)}$$

## Why Does the Theorem Work?

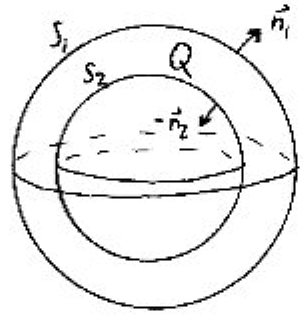
Roughly break $Q$ into cubes.




Again, FTC idea here - boundary!! Same flavor as Green, Stokes, FTCI. Larson 6ed → 1054

The only net change of flow throughout $Q$ will be at the boundary, $S$.
again, FTC idea here

## Extension




$$\iiint_Q (\operatorname{div} \vec{F})\, dV$$

$$= \iint_{S_1} \vec{F}\cdot\vec{n}_1\, dS + \iint_{S_2} \vec{F}\cdot(-\vec{n}_2)\, dS$$

Stewart 1137 5th ed E7 Used Thm. to prove Archimedes's principle