SECTIONS 18.6 AND 18.7: ADDITIONAL NOTES AND REVISIONS

SECTION 18.6: DIVERGENCE (GAUSS'S) THEOREM

Instead of doing my Example in my notes (#8), I will do the following Example:

Example

Find the flux of $\mathbf{F}(x,y,z) = \langle 2x, x^2z^3, 5z \rangle$ through any sphere S of radius 4.

Solution

$$\text{Flux} = \iint_S \mathbf{F} \bullet \mathbf{n}\, dS = \iiint_Q (\text{div } \mathbf{F})\, dV \text{, where}$$

$$\begin{aligned} \text{div } \mathbf{F} &= \frac{\partial}{\partial x}(2x) + \frac{\partial}{\partial y}(x^2z^3) + \frac{\partial}{\partial z}(5z) \\ &= 2 + 0 + 5 \\ &= 7 \end{aligned}$$

and Q is the region bounded by S.

$$\begin{aligned} \text{Flux} &= \iiint_Q 7\, dV \\ &= 7 \iiint_Q dV \\ &= 7(\text{Volume of } Q) \\ &= 7\left(\frac{4}{3}\pi(4)^3\right) \end{aligned}$$

since the volume of a sphere of radius r is $\frac{4}{3}\pi r^3$

$$\begin{aligned} &= 7\left(\frac{256\pi}{3}\right) \\ &= \frac{1792\pi}{3} \end{aligned}$$

SECTION 18.7: STOKES'S THEOREM

I will skip my Example (#6).

I may show in class why Green's Theorem is merely a special case of Stokes's Theorem.

We will make the usual assumptions for Stokes's Theorem. According to the theorem,

$$\text{Work } W = \oint_C \mathbf{F} \bullet \mathbf{T}\, ds = \iint_S (\mathbf{curl\ F}) \bullet \mathbf{n}\, dS$$

If S is a region of the xy-plane, we can call it R, and we use $\mathbf{n} = \mathbf{k}$:

$$\text{Work } W = \oint_C \mathbf{F} \bullet \mathbf{T}\, ds = \iint_R (\mathbf{curl\ F}) \bullet \mathbf{k}\, dA$$

This is called the <u>vector form of Green's Theorem</u>. Why?

Let $\mathbf{F}(x, y, 0) = \langle M(x, y), N(x, y), 0 \rangle$.

You will find that $(\mathbf{curl\ F}) \bullet \mathbf{k} = \dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}$.

We then have:

$$\begin{aligned} \text{Work } W &= \oint_C \mathbf{F} \bullet \mathbf{T}\, ds = \iint_R (\mathbf{curl\ F}) \bullet \mathbf{k}\, dA \\ &= \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA \end{aligned}$$

The last expression should look familiar!

“COMING FULL CIRCLE” IN CALCULUS

The Generalized Stokes’s Theorem covers all of the major vector calculus theorems in this chapter, as well as the classic Fundamental Theorem of Calculus (FTC) from Calculus I.

What did the FTC say?

If f is integrable on the interval $\left[a,b\right]$ with antiderivative F on that interval,

$$\begin{aligned}\int_a^b f\left(x\right)\,dx &= \left[F\left(x\right)\right]_a^b \\ &= F\left(b\right) - F\left(a\right)\end{aligned}$$

The FTC relates an integral over an interval to information at the endpoints (the “boundary”) of that interval.

In Chapter 18, we related a higher-dimensional integral over a region to a lower-dimensional integral over the boundary of the region.

Calculus III is a very natural extension of Calculus I!!