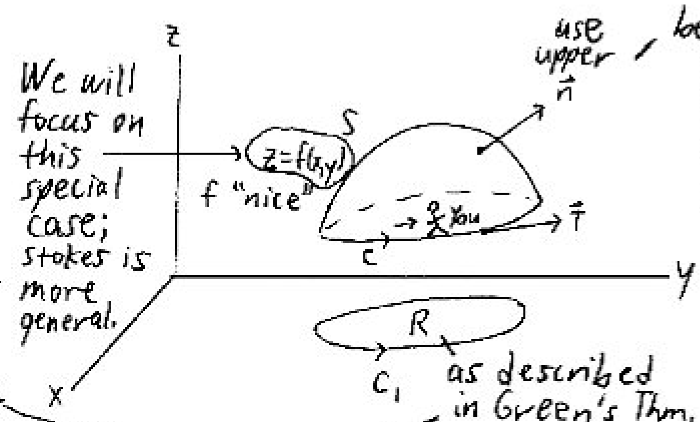
18.7: STOKES's THEOREM ("Green in 3D")

Stewart 1126 5ed ET

published in 1854 by George Stokes (Irish/Eng. mathem. physicist, 1819–1903).
Lord Kelvin also key in both thms.; wrote letter to Stokes in 1850.

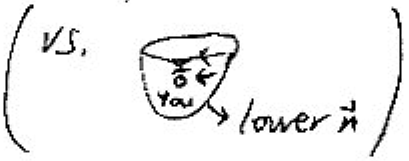
Broadly: Orientation, $\vec{n}$ consistent w/right-hand-rule idea.

We will focus on this special case; Stokes is more general.



R is "Green-type"
C_1: pw simple closed curve
$R = C_1 \cup$ interior of C_1

, because you must walk on top of C in the indicated direction in order to keep S to your left. This direction gives the positive orientation of C.



(See 18.5.6.)

$$\boxed{\text{Work} = \oint_C \vec{F}\cdot\vec{T}\,ds = \underbrace{\iint_S (\overrightarrow{\text{curl}}\,\vec{F})\cdot\vec{n}\,dS}_{\substack{=\text{ rotation of }\vec{F}\text{ about }\vec{n}\\ =\text{ rot }\vec{F}}} \;\Big|\; = \iint_{R_{xy}} (\overrightarrow{\text{curl}}\,\vec{F})\cdot\vec{\nabla}g\,dA}$$

where $g(x,y,z) = z - f(x,y)$

Assume $\vec{F}$ is "nice" in an open region containing S.

Other books: $\overrightarrow{\text{rot}}\,\vec{F} = \overrightarrow{\text{curl}}\,\vec{F}$

= Flux of $\overrightarrow{\text{curl}}\,\vec{F}$ across S

Swokowski → = Surface $\iint$ of normal component of $\overrightarrow{\text{curl}}\,\vec{F}$ over S.

Green's and Stokes's Thms. are extensions of FTC. So is Div. Thm.

This is the 3D extension of Green's Thm., which can be expressed as: $\oint_C \vec{F}\cdot\vec{T}\,ds = \iint_R (\overrightarrow{\text{curl}}\,\vec{F})\cdot \vec{k}\,dA$ in "vector form."

$\underbrace{\langle\ ,\ ,}_{\text{Given in }\mathbb{R}^2} 0\rangle$ ← Insert

Observe: If you think of "blowing a bubble" ($\perp$ to xy-plane) through C, you get the same value for the work integral!!

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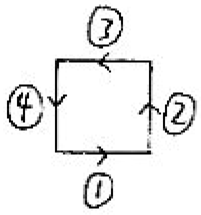
if $z=f(x,y)$ where f is "nice";
otherwise, more complicated story...

Stokes explains "paddlewheel" interpretation for $\overrightarrow{\text{curl}}$. (18.1.5)

Ex (#6)

If $\vec{F}(x,y,z) = \langle yz, xy, xz\rangle$, and C is the square with vertices $(0,0,2)$, $(1,0,2)$, $(1,1,2)$, and $(0,1,2)$, use Stokes' theorem to evaluate $\oint_C \vec{F}\cdot d\vec{r}$.

Without Stokes



$4 \int_s$

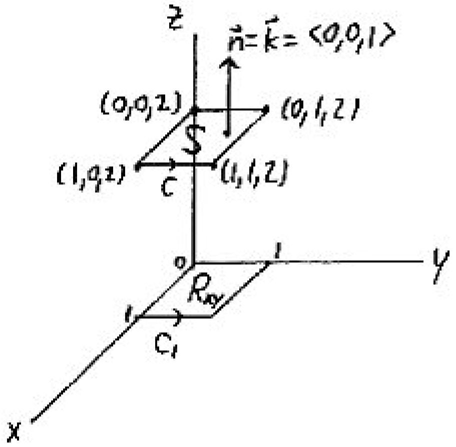
With Stokes

$$\overrightarrow{\text{curl}}\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & xy & xz \end{vmatrix}$$

$$= \left\langle \tfrac{\partial}{\partial y}(xz) - \tfrac{\partial}{\partial z}(xy),\ -\left[\tfrac{\partial}{\partial x}(xz) - \tfrac{\partial}{\partial z}(yz)\right],\ \tfrac{\partial}{\partial x}(xy) - \tfrac{\partial}{\partial y}(yz)\right\rangle$$

$$= \langle 0 - 0,\ -[z - y],\ y - z\rangle$$

$$= \langle 0, y-z, y-z\rangle$$



S lies on the graph of $z = 2$.

$$\underbrace{z - 2}_{g(x,y,z)} = 0$$

$$\vec{\nabla} g(x,y,z) = \langle 0,0,1 \rangle \quad (= \vec{k})$$

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_{R_{xy}} (\overrightarrow{\text{curl}}\, \vec{F}) \cdot \vec{\nabla} g \, dA$$

$$= \iint_{R_{xy}} \langle 0, y-z, y-z \rangle \cdot \langle 0,0,1 \rangle \, dA$$

$$= \iint_{R_{xy}} (y - \underset{\uparrow z=2}{z}) \, dA$$

<u>Note</u>: When $S = R_{xy}$, we end up with Green's Thm. Can you show this?

$$= \int_0^1 \int_0^1 (y-2) \, dx \, dy$$

$$= \underbrace{\left[\int_0^1 dx\right]}_{=1} \left[\int_0^1 (y-2) \, dy\right]$$

$$= \left[\frac{y^2}{2} - 2y\right]_0^1$$

$$= \left[\frac{(1)^2}{2} - 2(1)\right] - [0]$$

$$= \boxed{-\frac{3}{2}}$$

(Typed in review:)

When is $\vec{F}$ Conservative in $\mathbb{R}^3$? Equivalent statements:

In a connected region D (in which $\vec{F}$ is cont.)...
"in one piece"

Same as for $\mathbb{R}^2$ on 18.3.10, but

① $\vec{F}$ is conservative
(i.e., $\vec{F} = \vec{\nabla} f$ for some scalar potential func. f)

② We have IP: $\int_C \vec{F} \cdot d\vec{r}$

③ $\int_C \vec{F} \cdot d\vec{r} = 0$ for <u>every</u> simple closed curve C in D

replace (4a) → (4b) $\overrightarrow{\text{curl}}\ \vec{F} = \vec{0}$ throughout D
(i.e., $\vec{F}$ is <u>irrotational</u>)
if $\vec{F} = \langle M, N, P \rangle$ is "nice."

<u>Note</u>: $\int_C \vec{F} \cdot d\vec{r} = \int_C M\,dx + N\,dy + P\,dz$

If we start with (4b), then we require that D be <u>simply</u> connected.
different idea from $\mathbb{R}^2$ case

See pp. 1017-8.
This is s.c.:

A simple closed curve like this in D can be continuously deformed through D down to a pt. in D.

A donut (torus) is not.
There are simple closed curves in it that don't have capping surfaces.

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<u>Note</u> (4b) is a very natural extension of (4a) into $\mathbb{R}^3$!!

$$\overrightarrow{\text{curl}}\ \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & P \end{vmatrix}$$

$$= \left\langle \frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}, -\left(\frac{\partial P}{\partial x} - \frac{\partial M}{\partial z}\right), \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right\rangle$$

$$= \left\langle \frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right\rangle$$

Observe:

$$\overrightarrow{\text{curl}}\ \vec{F} = \vec{0} \iff \frac{\partial P}{\partial y} = \frac{\partial N}{\partial z},\ \frac{\partial M}{\partial z} = \frac{\partial P}{\partial x}, \text{ and } \underbrace{\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}}_{\text{from (4a) for } \mathbb{R}^2 \text{ !!}}$$

Also:

If $\vec{F}(x,y) = \langle M(x,y), N(x,y) \rangle$ is "nice" in $\mathbb{R}^2$, we can go to the $\mathbb{R}^3$ case by writing:

$$\vec{F}(x,y) = \langle M(x,y), N(x,y), \underset{\nwarrow \text{zero}}{0} \rangle$$

Then, $\overrightarrow{\text{curl}}\ \vec{F} = \langle 0, 0, \underbrace{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}_{\text{"scalar curl"}} \rangle$

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This is $\vec{0} \Leftrightarrow \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$