REVIEW: CH. 18

Here, we deal with Cartesian coordinates. (Schey and Marsden discuss other systems.)

VECTOR FIELDS (18.1)

We say a scalar function f is "nice" if its 1^{st}-order partial derivatives are continuous (and, therefore, f itself is) "where we care." A vector field **F** is "nice" if its components are nice.

$\mathbf{F}\big|_A$ is the vector associated with point A.

GRAD, CURL, AND DIV (18.1)

The del (or nabla) operator $\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \ldots \right\rangle$

$$\textbf{grad } f = \nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \ldots \right\rangle$$

(See Sections 16.6 and 16.7 for more on gradients.)

$$\textbf{curl } \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & P \end{vmatrix}$$

where $\mathbf{F}(x,y,z) = \langle M(x,y,z), N(x,y,z), P(x,y,z) \rangle$ is a vector field in $\mathbb{R}^3$.

$\left[\textbf{curl } \mathbf{F}\right]_A$ is a vector whose …

… direction indicates axis of rotation of field near P (use right-hand rule);
… length indicates strength of rotational effect.

These properties are justified by Stokes's Theorem in Section 18.7.

$$\text{div } \mathbf{F} = \nabla \bullet \mathbf{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \ldots \right\rangle \bullet \langle M, N, \ldots \rangle = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \ldots$$

where $\mathbf{F}(x, y, \ldots) = \langle M(x, y, \ldots), N(x, y, \ldots), \ldots \rangle$ is a vector field in $\mathbb{R}^n$.

$\left[\text{div } \mathbf{F}\right]_A$ is a scalar:

If this is negative, then there is a <u>sink</u> at A.
If this is positive, then there is a <u>source</u> at A.
(Think: Alphabetical order coincides with numerical order. Also, it measures the tendency of a fluid to diverge *from* A.)

If this is 0, then A has neither.

These properties are justified by the Divergence (or Gauss's) Theorem in Section 18.6.

<u>Warning</u>: Unlike **grad** and **curl**, div actually yields a scalar function, not a vector-valued function.

<u>Interesting Identities</u>

$\text{div}\left(\mathbf{curl}\ \mathbf{F}\right) = 0$, where $\mathbf{F}$ is in $\mathbb{R}^3$

$\mathbf{curl}\left(\mathbf{grad}\ f\right) = \mathbf{0}$, where f is a function of x, y, and z

<u>Technical Note</u>: We assume that f and the components of $\mathbf{F}$ have continuous 2^{nd}-order partial derivatives.

<u>LINE / PATH INTEGRALS (18.2)</u>

We will integrate along a piecewise-smooth ("ps") path C.
A ps path has no breaks, corners, cusps, or backtracks.
Arrowheads indicate orientation along the path.

Here, we will focus on formulas used for the 2D xy-plane.
The formulas below are naturally extended to the 3D xyz case.

Let's say C is parameterized by: $\mathbf{r}(t) = \langle x(t), y(t) \rangle$

We may then rewrite problems in terms of t alone instead of both x and y.

<u>Technical Note</u>: The smoothness condition requires that the tangent VVF $\mathbf{r}'(t)$ be non-$\mathbf{0}$ along the path (except possibly at endpoints) and continuous along the path. For ps curves, replace "path" with "pieces of the path."

ds, the differential of arc length, can be expressed in many ways:

$$ds = \sqrt{(dx)^2 + (dy)^2} \qquad \text{(from Pythagorean Theorem)}$$

$$= \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \quad (\text{if } t \text{ is increasing, and, therefore, } dt > 0)$$

Warning: For now, let's say we are required to parameterize paths (or pieces of paths) in such a way that t is increasing consistently with the orientation. Otherwise, we replace dt with $|dt|$ or $-dt$ in these formulas.

Because of this issue, we typically avoid using something like $\int_2^1$ in our initial formulas (though they may appear after using, say, u-substitutions); we would want to parameterize so that the higher number is always on top. This requirement will be removed later on, when we deal with integrals involving vector fields.

Recall that $\mathbf{r}(t) = \langle x(t), y(t) \rangle$, so $\mathbf{r}'(t) = \mathbf{v}(t) = \left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle$, and we have:

$$ds = \|\mathbf{r}'(t)\|\, dt$$

$$= \|\mathbf{v}(t)\|\, dt$$

"Infinitesimal" Idea: (distance covered) = (speed) $\times$ (change in time)

<u>Line / Path Integral</u>: $\int_C f(x,y)\,ds$

<u>Examples / Applications</u>

<u>Lateral Surface Area</u>

Case: f gives the height of a "wall" built upon C.
We require: $f(x,y) \geq 0$ on C.

<u>Arc Length of C</u>

Case: $f(x,y) = 1$

$$L = \int_C ds = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\; dt$$

where $t = a$ corresponds to the initial endpoint of C,
$t = b$ corresponds to the terminal endpoint of C, and
$a < b$.

<u>Mass of C</u>

Case: $f(x,y) = \delta(x,y)$, linear mass density

$$m = \int_C \delta(x,y)\,ds$$

There are various acceptable ways to smoothly parameterize [pieces of] C.

Know how to parameterize pieces of circles, ellipses, lines, etc.

A Work Integral as a Line Integral of a ["Nice"] Vector Field, **F**

Let W = the work done by **F** on a particle moving along C (in the direction of orientation). Think: How much does **F** help out?

$$W = \int_C \mathbf{F} \bullet \mathbf{T}\, ds$$

Note: $\mathbf{F} \bullet \mathbf{T}$ is a scalar function representing the tangential component of **F** along C. It may be considered a very special case of $f(x, y)$.

Note (on the parameterization of the motion of the particle): The particle's speed is still irrelevant to the value of W, but reversing the orientation (meaning that we replace C with $-C$) changes the sign of W (if it is nonzero):

$$\int_C \mathbf{F} \bullet \mathbf{T}\, ds = -\int_{-C} \mathbf{F} \bullet \mathbf{T}\, ds$$

This is because **T** in the second integral is actually $-(\mathbf{T}$ in the first integral$)$, because the orientation is reversed. It is convenient (though somewhat sloppy) to retain the **T** notation.

Surprise: In these work problems (and the like), something like $\int_2^1$ **may** be permitted in our initial formulas, so long as **T** is directed appropriately. This is a change from before.

We did not encounter this sign flip in the previously mentioned surface area, arc length, mass problems, and the like, which makes geometric sense. For those problems:

$$\int_C f(x, y)\, ds = \int_{-C} f(x, y)\, ds$$

Technical Note: Remember that, when ds is unraveled in those problems, we may need to use $|dt|$, which would equal $-dt$ if t is decreasing and, thus, $dt < 0$.

Different Ways of Expressing a Work Integral

$$W = \int_C \mathbf{F} \bullet \mathbf{T}\, ds$$

$$W = \int_C \mathbf{F} \bullet \frac{\mathbf{r}'(t)}{\cancel{\|\mathbf{r}'(t)\|}} \cancel{\|\mathbf{r}'(t)\|}\, dt$$

$$W = \int_C \mathbf{F} \bullet \mathbf{r}'(t)\, dt$$

The above may be the best formula if $\mathbf{F}$ and $\mathbf{r}$ are given in terms of t, in which case $\mathbf{F}$ may only be known for points along C.

Let's now play with the notation: $\mathbf{r}'(t)\, dt = \frac{d\mathbf{r}}{dt}\, dt = d\mathbf{r}$

$$W = \int_C \mathbf{F} \bullet d\mathbf{r}$$

The above is our "shortest" formula.

Let's say we have:

$\mathbf{F} = \langle M(x,y), N(x,y) \rangle$, continuous in a region containing C

$d\mathbf{r} = \langle dx, dy \rangle$

$$W = \int_C \langle M(x,y), N(x,y) \rangle \bullet \langle dx, dy \rangle$$

$$W = \int_C M\, dx + N\, dy$$

This last form, differential form, is often used in problems.

It is OK to have something like $\int_2^1$ in your initial formulas.
For example, in the second formula, if t is decreasing, then we may want to replace dt with $-dt$. However, $\mathbf{r}'(t)$ would also be replaced by $-\mathbf{r}'(t)$, so we have a "double negative" that preserves the correctness of that formula.

Sections 18.3, 18.4, and 18.7 offer shortcuts for computing W in special cases.

INDEPENDENCE OF PATH (IP) (18.3)

Throughout Ch.18, we assume that D is a connected (i.e., "one-piece") region in which **F** is continuous and that C is a ps path in D.

What does it mean to have IP for **F** in D?

The value of $W = \int_C \mathbf{F} \bullet d\mathbf{r}$ is the same for <u>any</u> ps path in D that has the same initial point (A) and terminal point (B) as C does. We can then write:

$$W = \int_C \mathbf{F} \bullet d\mathbf{r} = \int_A^B \mathbf{F} \bullet d\mathbf{r}$$

To compute W, we can then:

- Choose an "easier" path in D and use 18.2 methods, or …

- Find a potential function f (such that $\mathbf{F} = \nabla f$) and apply the Fundamental Theorem for Line Integrals (FTLI):

$$W = \int_C \mathbf{F} \bullet d\mathbf{r} = \int_A^B \mathbf{F} \bullet d\mathbf{r} = \left[f \right]_A^B = f\big|_B - f\big|_A$$

- If C is a closed curve, then we can make $A = B$, and we have:

$$W = \int_C \mathbf{F} \bullet d\mathbf{r} = 0$$

<u>WHEN IS F CONSERVATIVE? EQUIVALENT STATEMENTS (18.1 / 18.3 / 18.7):</u>

In a connected region D (in which **F** is continuous) …

1) **F** is conservative (i.e., $\mathbf{F} = \nabla f$ for some scalar potential function f)

2) We have IP: $\int_C \mathbf{F} \bullet d\mathbf{r}$

3) $\int_C \mathbf{F} \bullet d\mathbf{r} = 0$ for <u>every</u> simple closed curve C in D

   A simple closed curve only self-intersects at one point, which is both its initial point (A) and its terminal point (B).

4a) $\dfrac{\partial N}{\partial x} = \dfrac{\partial M}{\partial y}$ throughout D (if **F** is "nice" in $\mathbb{R}^2$)

   where $\mathbf{F} = \langle M, N \rangle$, in which case:

$$\int_C \mathbf{F} \bullet d\mathbf{r} = \int_C M\,dx + N\,dy$$

   If we start with statement 4a), then we require that D be <u>simply</u> connected, meaning it is in "one piece" and has no holes.

4b) **curl F** = **0** throughout D (if **F** is "nice" in $\mathbb{R}^3$)

   In other words, **F** is <u>irrotational</u> [in a local sense].

   If we start with statement 4b), then we require that D be <u>simply</u> connected. See pp.1017-8 in Swokowski and my Notes 18.7.4 for a definition of simply connected regions in $\mathbb{R}^3$.

   You may consider 4a) to be a special case of 4b) in which $\mathbf{F}(x, y) = \langle M(x, y), N(x, y), 0 \rangle$.

<u>Note</u>: Inverse square fields such as those used to study gravity and electromagnetism are conservative. (See my Notes 18.1.8)

GREEN'S THEOREM (18.4)

Assume:

We are in $\mathbb{R}^2$.

C is a ps simple **closed** curve that forms the boundary of R, which is a closed subset of an open region D.

C is oriented in the positive direction – this is indicated by $\oint_C$ – meaning that R is always on the left as we look down on the xy-plane and travel along C in this direction.

$\mathbf{F}(x, y) = \langle M(x, y), N(x, y) \rangle$ in $\mathbb{R}^2$, where M and N are "nice" in D.

Then,

$W = \oint_C \mathbf{F} \bullet d\mathbf{r} = \oint_C M\, dx + N\, dy$ equals, by Green's Theorem,

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA$$

In fact,

$$\text{Area of } R = \frac{1}{2} \oint_C -y\, dx + x\, dy = \oint_C -y\, dx = \oint_C x\, dy$$

It may be easier to remember: Area of $R = \dfrac{1}{2} \oint_C x\, dy - y\, dx$

If R is not simply connected (i.e., it has holes), then try slitting R. Make sure orientations remain positive.

SURFACE INTEGRALS (18.5)

Assume: f is "nice" (i.e., is continuous and has continuous 1st-order partial derivatives "where we care.")

Let S be the graph of $z = f(x, y)$. It corresponds to a "projection region" R_{xy} in the xy-plane. We have analogous formulas when S is the graph of $y = f(x,z)$ or of $x = f(y,z)$.

You may need to take a given equation and solve for z, for example.

Surface Area of S $= \iint_S dS$

$$= \iint_{R_{xy}} \sqrt{1 + \left[f_x(x,y)\right]^2 + \left[f_y(x,y)\right]^2}\ dA$$

Mass of S $= \iint_S \underbrace{\delta(x,y,z)}_{\substack{\text{Area mass}\\ \text{density}}}\ dS$

$$= \iint_{R_{xy}} \delta(x,y,f(x,y))\sqrt{1 + \left[f_x(x,y)\right]^2 + \left[f_y(x,y)\right]^2}\ dA$$

Flux (Flow) of a Vector Field, **F**, across *S*:

$\iint\limits_S \mathbf{F}\bullet\mathbf{n}\,dS$, where

$g(x,y,z) = z - f(x,y)$

(Observe that $z = f(x,y) \Leftrightarrow \underbrace{z - f(x,y)}_{g(x,y,z)} = 0$, so *S* is a level surface of g)

and

$\mathbf{n} = \dfrac{\nabla g}{\|\nabla g\|} = \dfrac{\langle -f_x, -f_y, 1\rangle}{\sqrt{1+(f_x)^2+(f_y)^2}}$, the upper unit normal to *S*.

Note: $\mathbf{F}\bullet\mathbf{n}$ is a scalar function representing the normal component of **F** as we sweep over *S*.

Note: If *S* is a closed surface, we take the unit outer normal.

$$\iint\limits_S \mathbf{F}\bullet\mathbf{n}\,dS = \iint\limits_{R_{xy}} \mathbf{F}\bullet\frac{\langle -f_x, -f_y, 1\rangle}{\cancel{\sqrt{1+(f_x)^2+(f_y)^2}}}\cancel{\sqrt{1+(f_x)^2+(f_y)^2}}\,dA$$
$$= \iint\limits_{R_{xy}} \mathbf{F}\bullet\nabla g\,dA$$

We often need to replace z with $f(x,y)$.

DIVERGENCE (OR GAUSS'S) THEOREM (18.6)

This can help us compute the flux across a closed surface.

Let S be a closed surface bounding a 3D region Q.
Let $\mathbf{n}$ be the unit outer normal to S.
Let $\mathbf{F}$ be a vector field in $\mathbb{R}^3$ that is "nice" throughout Q.

Then,

$$\text{Flux} = \iint_S \mathbf{F} \bullet \mathbf{n}\, dS = \iiint_Q \left(\text{div}\, \mathbf{F}\right) dV$$

This leads to the interpretation of div in terms of sinks and sources on Quiz 5 – R2.

Note: In 2D, by Green's Theorem, if C is the kind of simple closed curve described in Green's Theorem, then:

$$\text{Flux} = \oint_C \mathbf{F} \bullet \mathbf{N}\, ds = \iint_R \left(\text{div}\, \mathbf{F}\right) dA$$

Note: This can be extended to regions with holes inside, provided $\mathbf{n}$ is always chosen correctly.

STOKES'S THEOREM (18.7)

This can help us compute a work integral by using a surface integral.
Think: Green in 3D.

Assume:

S has equation $z = \underbrace{f\left(x, y\right)}_{\text{"nice"}}$ and is a capping surface for a ps simple closed curve C.

F is "nice" throughout an open region containing S.

The projection of C in the xy-plane, C_1, bounds a region R as described in Green's Theorem.

We take the positive orientation of C and the choice of the unit normal **n** to be the ones corresponding to a "walk" in which S always lies to your left.

Then,

$$\text{Work } W = \oint_C \mathbf{F} \bullet \mathbf{T}\, ds = \iint_S \left(\textbf{curl F}\right) \bullet \mathbf{n}\, dS$$

(This is the flux of **curl F** across S.)
(This is the surface integral of the normal component of **curl F** over S.)

If we adopt the setup in 18.5, this equals: $\iint_R \left(\textbf{curl F}\right) \bullet \nabla g\, dA$

Given C, observe that different choices may be made for the capping surface S.
("Bubble blowing")

This leads to the "paddlewheel" interpretation of **curl** on Quiz 5 – R1.
(See pp.1013-4 in Swokowski.)

GENERAL OBSERVATIONS AND TRICKS

When integrating, it may help to go to polar, cylindrical, or spherical coordinates.

Symmetry can be a useful tool, but make sure you can use it!

Check the integrand and the region of integration for symmetry.
You may be able to rewrite an integral using "0"s as limits of integration.

If $f(x, y)$ is symmetric in x and y, then f_x and f_y will have analogous forms.

Green's Theorem is used to compute the <u>work</u> integral $\oint_C \mathbf{F} \bullet \mathbf{T}\, ds$ for a simple closed <u>curve</u> C, while the Divergence (Gauss's) Theorem is used to compute the <u>flux</u> integral $\iint_S \mathbf{F} \bullet \mathbf{n}\, dS$ for a closed <u>surface</u> S.

Both Green's Theorem and Stokes's Theorem allow us to use double integrals to compute the <u>work</u> integral $\oint_C \mathbf{F} \bullet \mathbf{T}\, ds$ along a simple closed <u>curve</u> C. However, the settings are different: $\mathbb{R}^2$ vs. $\mathbb{R}^3$. Stokes's Theorem, which states:

$$\oint_C \mathbf{F} \bullet \mathbf{T}\, ds = \iint_S (\mathbf{curl\ F}) \bullet \mathbf{n}\, dS$$

may be viewed as a 3D extension of the "2D" Green's Theorem, which can be expressed in "vector form" as:

$$\oint_C \mathbf{F} \bullet \mathbf{T}\, ds = \iint_R (\mathbf{curl\ F}) \bullet \mathbf{k}\, dA.$$