

CH. 1: SYSTEMS OF LINEAR Eqs.L1: INTRO① Linear Eqs.Standard Form

$$a_1 x_1 + a_2 x_2 + \dots + a_n x_n = b$$

where " $a_i$ "'s are real coeffs.

" $x_i$ "'s are variables

" $b$ " is a real #

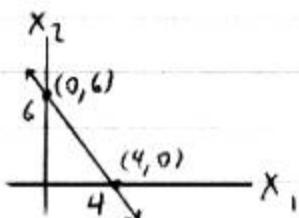
$$\text{Ex } 3x_1 + 2x_2 = 12$$

is a linear eq. in  $x_1$  and  $x_2$ .

$$a_1 = 3$$

$$a_2 = 2$$

$$b = 12$$



Before you'd say  
big deal - it's a  
line  
(Cartesian)

Every pt. represents  
a solution.  
∞ many sol'n's making  
up the solution set.

## (8) Parametric Representation of a Solution Set

Describe the sol'n set.

$$\text{Ex Solve } 3x_1 + 2x_2 = 12$$

Solve for  $x_1$  (say)

$$3x_1 = -2x_2 + 12$$

$$\underset{\uparrow}{x_1} = -\frac{2}{3}\underset{\uparrow}{(x_2)} + 4$$

depends free variable  
on  $x_2$  (plug in)

Let  $x_2 = t$   
("parameter")

Let's say I  
have a (boring)  
friend, and he  
wants to know  
some sol'n's to  
this eq. Here's  
a recipe for  
producing sol'n's.  
Plug in the into  
 $3, -5, \pi$

Sol'n set:

$$\begin{aligned} x_1 &= -\frac{2}{3}t + 4 \\ x_2 &= t \end{aligned}$$

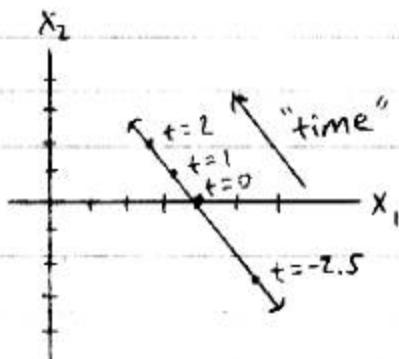
$t$  is any real #

## Table of particular sol'n's

Do  $x_2$  1st.

what does  
 $t$  often stand  
for?

$\frac{t}{0}$	$\frac{x_1}{4}$	$\frac{x_2}{0}$	$\rightarrow (4, 0)$
1	$3\frac{1}{3}$	1	
2	$2\frac{2}{3}$	2	
-2.5	$5\frac{2}{3}$	-2.5	



Ex. Solve  $2x_1 + 3x_2 - 4x_3 = 1$

Solve for  $x_1$

$$2x_1 = -3x_2 + 4x_3 + 1$$

$$x_1 = -\frac{3}{2}(x_2) + 2(x_3) + \frac{1}{2}$$

↑  
2 free  
vars.

Let  $x_2 = t$   
 $x_3 = u$

Sol'n set:

$$\begin{aligned}x_1 &= -\frac{3}{2}t + 2u + \frac{1}{2} \\x_2 &= t \\x_3 &= u\end{aligned}$$

$t, u$  are any real #s

Table

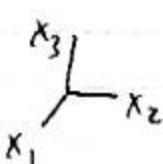
Do  $x_2, x_3$  1st.

$t$	$u$	$x_1$	$x_2$	$x_3$	
0	0	$\frac{1}{2}$	0	0	①
1	0	-1	1	0	②
1	1	1	1	1	③

$$(-3, 5\frac{1}{2})$$

As you vary  
values for  $t, u$ ,  
you sweep out  
an entire plane  
of solns.

As you vary  $t, u$ ,  
you sweep out  
pts. on this line seg.

In  plane of solns



### ③ Solving Systems of Linear Eqs.

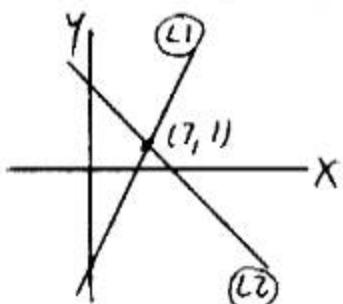
Ex Solve  $\begin{cases} 3x - 2y = 19 & (L1) \\ x + y = 8 & (L2) \end{cases}$

A solution to this system solves all the eqs.

### Method 1 (Graphing)

This line gives all solns to this eq...

If the sol'n is  $(\frac{2}{3}, \frac{17}{3})$



An intersection point is a solution. (2P)

### Method 2 (Substitution)

We solve for one of the vars in one of the eqs.

$$\begin{cases} 3x - 2y = 19 \\ x + y = 8 \end{cases} \Rightarrow y = 8 - x$$

$$3x - 2(8 - x) = 19$$

$$x = 7$$

$$y = 8 - x = 8 - 7 = 1$$

$$\begin{cases} x = 7 \\ y = 1 \end{cases} \text{ or } \{(7, 1)\}$$

sol'n set  
consisting of  
1 sol'n, an  
ordered pair

### Method 3 (Addition Method)

When we add  
equals to  
equals, we  
get equals.

$$\left\{ \begin{array}{l} 3x - 2y = 19 \\ x + y = 8 \end{array} \right. \leftarrow \cdot 2$$

$$\left\{ \begin{array}{l} 3x - 2y = 19 \\ 2x + 2y = 16 \end{array} \right. \quad \begin{array}{l} \downarrow \text{Add} \\ \downarrow \text{eqs.} \end{array}$$

$$\begin{array}{rcl} 5x & = & 35 \\ x & = & 7 \end{array}$$

$$\rightarrow 7 + y = 8$$

$$\quad \quad \quad \begin{array}{l} y = 1 \end{array}$$

$\{(7, 1)\}$

More methods later!

## ① Types of Solution Sets

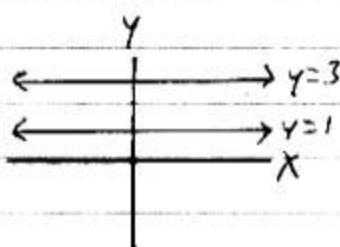
For systems in  $\mathbb{R}^x$

# of sol'n's = # of intersection pts.

When do  
2 lines in  
the xy-plane  
never intersect?

Ex (0 sol'n's)

$$\begin{cases} y = 1 \\ y = 3 \end{cases}$$



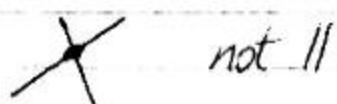
Different parallel (II)  
lines never intersect.

Sol'n set =  $\emptyset$  (empty/null set)

We can't  
reconcile  
the eqs.

The system is inconsistent.

Ex (1 sol'n)

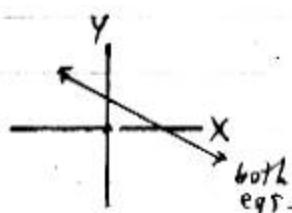


How can 2  
lines in  
the xy-plane  
intersect each  
other  $\infty$  many  
times?

Ex ( $\infty$  many sol'n's)

$$\begin{cases} x + 2y = 5 \\ 2x + 4y = 10 \end{cases}$$

same stupid line!



The equations are dependent.

To write sol'n set:

Parametric rep.

$$\begin{aligned}x + 2y &= 5 \\x &= 5 - 2y\end{aligned}$$

$y$  free

Let  $y = t$

$$\begin{aligned}x &= 5 - 2t \\y &= t \\t &\text{ is any real #}\end{aligned}$$

### Summary

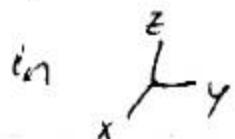
# sol'n's	Ex	System is	
0	II	inconsistent	Eqs. are ↓ independent
1	→		
∞ many	↓ both	consistent	dependent

↑  
Only possibls.  
for systems  
of linear eqs.

## ⑥ Systems of 3 Linear Eqs. in 3 Variables

Ex  $\begin{cases} x + 2y - z = -3 \\ 2x - y + 3z = 4 \\ z = 3 \end{cases}$  → plane

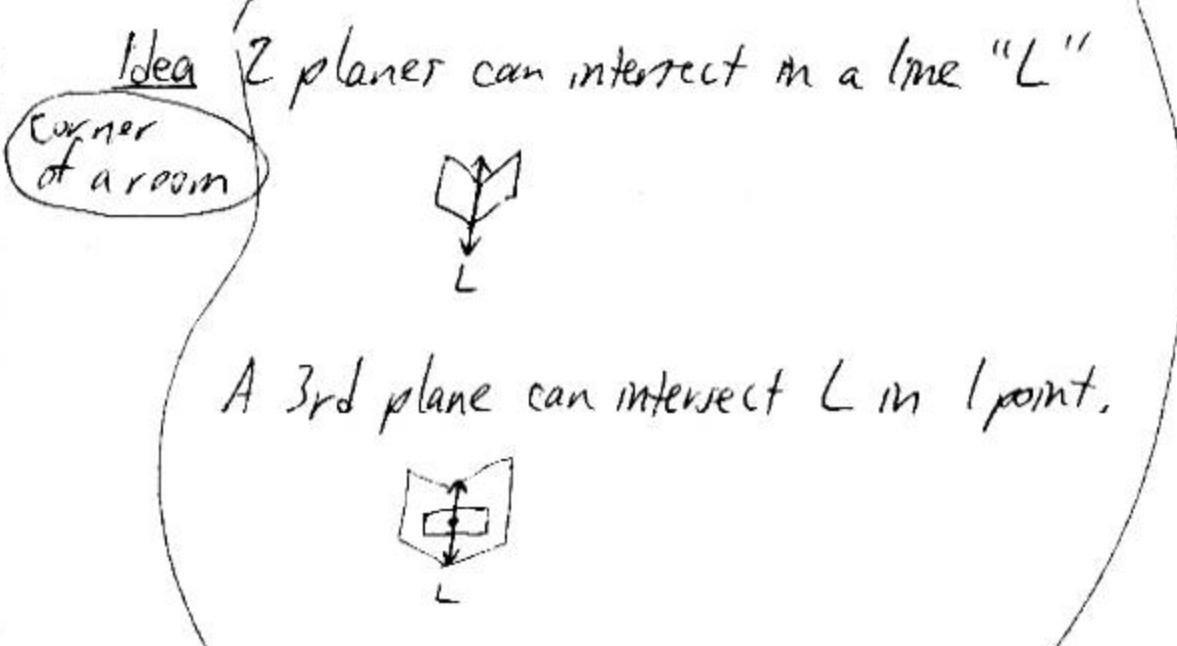
(The graph of each eq. is a plane)



1 sol'n:  $(-\frac{1}{2}, 1, 3)$

All 3 planes intersect here.  
A solution must lie on all 3 planes.

In a hospital  
researcher sees  
a fly on a wall  
sick, nothing  
better to do,  
travelling  
movement  
relative to  
the walls.



What  
if only  
2 planes  
are II?

Exs (0 sol'n) (no point on all 3)

Ex (oo many sol'n) (same stupid)  
plane

10 line of sol'n's  
in 3-space

20 plane of sol'n's  
in 3-space

## 1.2: MATRIX METHODS FOR SOLVING SYSTEMS

### ① Matrices

A matrix is a box of #s.

Ex A  $2 \times 3$  matrix:

size

$$\begin{array}{c} 2 \\ \text{rows} \rightarrow \\ \left[ \begin{array}{ccc} 1 & 0 & 2 \\ 0 & 1 & 5 \end{array} \right] \\ \uparrow \quad \uparrow \quad \uparrow \\ 3 \text{ columns} \end{array}$$

has 6 entries/elements

Form for an  $m \times n$  matrix:

$$\begin{array}{c} m \\ \text{rows} \rightarrow \\ \left[ \begin{array}{cccc} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & \ddots & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{array} \right] \\ \uparrow \quad \uparrow \quad \uparrow \\ n \text{ columns} \end{array}$$

has  $mn$  entries: the " $a_{ij}$ "'s  
 row col.

square"order"

$$n \begin{bmatrix} 1 & n \end{bmatrix}$$

### (B) Writing the Augmented Matrix for a System

① Write the eqs. in standard form

Ex  $\begin{cases} 3x + 2y - z = 0 \\ -2x + y = 3 \\ y = 4z - 5 \end{cases}$  ✓

(OK that coeff. of  $x$  is "-")

② Line up like terms (in columns)

$$\begin{cases} 3x + 2y - z = 0 \\ -2x + y = 3 \\ y = 4z - 5 \end{cases}$$

$\begin{pmatrix} \uparrow & \uparrow & \uparrow \\ x & y & z \end{pmatrix}$   
 terms

③ (Optional). Write in 0, ±1 coeffs.

$$\begin{pmatrix} +y \rightarrow +1y \\ -z \rightarrow -1z \\ \text{no } x \text{ term} \rightarrow 0x \end{pmatrix}$$

$$\begin{cases} 3x + 2y - 1z = 0 \\ -2x + 1y + 0z = 3 \\ 0x + 1y - 4z = -5 \end{cases}$$

1.2.3

(4) Write the augmented matrix

Even though it's tempting to do it by columns, you may be better off doing it by rows. When you read the eqs., you're less likely to omit the +1s. Still, lining up like terms a good idea.

			coeffs. of	Right-hand side. (RHS)
$\frac{x}{3}$	$\frac{y}{2}$	$\frac{z}{-1}$		0
-2	1	0		3
0	1	-4		-5

coefficient  
 matrix  
 (book  
 omits)

skip

Ex  $\left\{ \begin{array}{l} 3x - y = -4 \\ 5x = 7 \end{array} \right.$

$\left\{ \begin{array}{l} 3x - 1y = -4 \\ 5x + 0y = 7 \end{array} \right.$

$$\left[ \begin{array}{ccc|c} 3 & -1 & | & -4 \\ 5 & 0 & | & 7 \end{array} \right]$$

## ⑥ Elementary Row Operations (EROs)

Equivalent equations have the same sol'n set.

$$\text{Ex } \begin{array}{l} 2x - 1 = 5 \\ 2x = 6 \\ x = 3 \end{array} \quad \begin{array}{l} \downarrow \text{sequence} \\ \text{of eqs.} \\ \leftarrow \text{an eq. whose} \\ \text{sol'n is obvious} \end{array}$$

Equivalent systems have the same sol'n set.

EROs: legal  
chess moves

EROs allow us to write a sequence of equivalent systems until we get one whose sol'n is easy to find.

EROs:

### ① Row Interchange

If we have a sol'n,  
what does it do?

$$\text{Ex } \left\{ \begin{array}{l} 3x - y = 1 \\ x + y = 4 \end{array} \right. \Rightarrow \left\{ \begin{array}{l} x + y = 4 \\ 3x - y = 1 \end{array} \right. \quad \begin{array}{l} (\text{same}) \\ (\text{sol'n set}) \end{array}$$

$$\begin{array}{l} R_1 \rightarrow [3 \quad -1 \quad | \quad 1] \\ R_2 \rightarrow [1 \quad 1 \quad | \quad 4] \end{array} \sim \begin{array}{l} [1 \quad 1 \quad | \quad 4] \\ [3 \quad -1 \quad | \quad 1] \end{array}$$

$R_1 \leftrightarrow R_2$

(We can rewrite the rows in any order.)

### ② Row Rescaling

We can <sup>mult.</sup> (or  $\div$ ) through a row by any (non-0) #.

$$\text{Ex } \begin{cases} \frac{1}{2}x + \frac{1}{2}y = 3 \\ y = 4 \end{cases} \left( \rightarrow \begin{cases} x + y = 6 \\ y = 4 \end{cases} \right)$$

$$\cdot 2 \rightarrow \left[ \begin{array}{cc|c} \frac{1}{2} & \frac{1}{2} & 3 \\ 0 & 1 & 4 \end{array} \right] \sim \left[ \begin{array}{cc|c} 1 & 1 & 6 \\ 0 & 1 & 4 \end{array} \right]$$

$$(2 \cdot R_1 \rightarrow R_1 \text{ (new)})$$

### ③ Row Replacement

We can add a multiple of one row to another row.

(Idea: Addition Method.)

It turns out we want to turn the " $-2$ " into a "0".

$$\text{Ex } \begin{cases} x + 3y = 3 & \leftarrow (2) \\ -2x + 5y = 16 \end{cases}$$

$$\begin{cases} 2x + 6y = 6 & \leftarrow \text{revert to original} \\ -2x + 5y = 16 \end{cases}$$

$$\begin{cases} 11y = 22 & \leftarrow \text{can replace} \\ (\text{hybrid eq.}) \end{cases}$$

$$\begin{cases} x + 3y = 3 \\ 11y = 22 \end{cases}$$

$$\begin{array}{c} R_1 [1 \ 3 \ | \ 3] \\ R_2 [-2 \ 5 \ | \ 16] \end{array}$$

want 0

$$(R_2 + 2 \cdot R_1 \rightarrow R_2 \text{ (new)})$$

$$\begin{array}{r} R_2: -2 \ 5 \ | \ 16 \\ + 2 \cdot R_1: 2 \ 6 \ | \ 6 \\ \hline \text{new } R_2: 0 \ 11 \ | \ 22 \end{array} \quad \downarrow \text{Add}$$

$$\begin{array}{l} \text{Same } R_1 \\ \text{New } R_2: 0 \ 11 \ | \ 22 \\ \left[ \begin{array}{cc|c} 1 & 3 & 3 \\ 0 & 1 & 2 \end{array} \right] \leftarrow \div 11 \\ \left[ \begin{array}{cc|c} 1 & 3 & 3 \\ 0 & 1 & 2 \end{array} \right] \text{ Good form!} \end{array}$$

Only ~4%  
have seen  
-

Meyer p.1  
Chinnere-Zeta BC  
+ Europe  
Gauss popularized  
it.

1.2.6

## ① Gaussian Elimination (Easy Exs.)

Method for solving systems.

Steps

① Write the augmented matrix.

② Use EROs to write a sequence of row-equivalent matrices until you get the form

$$\left[ \begin{array}{cccc|c} 1 & \cdot & \cdot & \cdot & \\ 0 & 1 & \cdot & \cdot & \\ \vdots & & \ddots & & \\ 0 & 0 & 0 & 1 & \end{array} \right]$$

All "0"s below      "1's along the main diagonal  
(it coeff. matrix is square)

Strategy: "Correct" the columns from left-to-right.

③ Write the new system.

④ Use back-substitution to find the solution.

Can ✓ in original system.

what?  
checkmate?

Top of a  
jigsaw puzzle  
book.

You don't want  
to undo the  
good work  
you've done.

Before you  
get your  
dirty hands  
all over it...

Ex Lia 24 Solve  

$$\begin{cases} 3x - 5y = 7 \\ x - y = 1 \end{cases}$$

$$\left( \begin{array}{l} \begin{cases} 3x - 5y = 7 \\ x - y = 1 \end{cases} \text{ (optional)} \end{array} \right)$$

Augmented Matrix  
want "1"

$$\left[ \begin{array}{ccc|c} 3 & -5 & | & 7 \\ 1 & -1 & | & 1 \end{array} \right] \leftarrow \text{If we } \div 3, \text{ fractions!}$$

Let's  $R_1 \leftrightarrow R_2$  (Write steps - in case you  $\checkmark$  later.)

$$\left[ \begin{array}{ccc|c} 1 & -1 & | & 1 \\ 3 & -5 & | & 7 \end{array} \right] \leftarrow \text{can't } \cdot 0$$

want "0"  
(elimination)

Let's:

$$\left( \begin{array}{l} R_2 - 3 \cdot R_1 \rightarrow R_2 \\ (\text{old}) \qquad \qquad \qquad (\text{new}) \end{array} \right)$$

$$R_2 + (-3) \cdot R_1$$

$$\begin{array}{r} R_2: 3 \quad -5 \quad | \quad 7 \\ +(-3) \cdot R_1: -3 \quad 3 \quad | \quad -3 \\ \hline \text{new } R_2: 0 \quad -2 \quad | \quad 4 \end{array} \quad \text{Add}$$

$$\left[ \begin{array}{ccc|c} 1 & -1 & | & 1 \\ 0 & -2 & | & 4 \end{array} \right] \leftarrow \div(-2) \text{ or } \cdot(-\frac{1}{2})$$

want "1" here

Don't want to use this row - we'll disturb "0"

1.2.8

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \end{bmatrix}$$

⇒ System

$$\left( \begin{array}{l} l_1 x - l_1 y = 1 \\ 0x + l_2 y = -2 \end{array} \right) \quad \begin{array}{l} (\text{can}) \\ (\text{skip}) \end{array}$$

$$\left\{ \begin{array}{l} x - y = 1 \\ y = -2 \end{array} \right. \quad \uparrow$$

Back-Sub

$$y = -2$$

(You)

$$\begin{aligned} x - y &= 1 \\ x - (-2) &= 1 \\ x + 2 &= 1 \\ x &= -1 \end{aligned}$$

What's my  
sol'n?

$$\left\{ \begin{array}{c} x \\ y \end{array} \right. \left\{ (-1, -2) \right\}$$

Can ✓ (in original system.)

up to 23,  
except  
7, 4, 11, 21

Ex Solve  $\begin{cases} 3x_1 + 2x_2 + x_3 = 8 \\ 6x_1 - x_2 + 3x_3 = 18 \\ -9x_1 + x_2 - x_3 = -20 \end{cases}$

Like rusty #32

$$\begin{cases} 3x_1 + 2x_2 + x_3 = 8 \\ 6x_1 - x_2 + 3x_3 = 18 \\ -9x_1 + x_2 - x_3 = -20 \end{cases}$$

$$\left( \begin{cases} 3x_1 + 2x_2 + 1x_3 = 8 \\ 6x_1 - 1x_2 + 3x_3 = 18 \\ -9x_1 + 1x_2 - 1x_3 = -20 \end{cases} \text{ (optional)} \right)$$

mults of 3

use "0"  
want "0"

3	2	1	8
6	-1	3	18
-9	1	-1	-20

(If we ÷ 3, fractions early!)

want  $\frac{1}{0}$

$$\begin{array}{r}
R_2: 6 \quad -1 \quad 3 \quad | \quad 18 \\
+ (-2) \cdot R_1: -6 \quad -4 \quad -2 \quad | \quad -16 \\
\hline
\text{new } R_2: 0 \quad -5 \quad 1 \quad | \quad 2
\end{array} \quad \downarrow \text{Add}$$

$$\begin{array}{r}
R_3: -9 \quad 1 \quad -1 \quad | \quad -20 \\
+ 3 \cdot R_1: 9 \quad 6 \quad 3 \quad | \quad 24 \\
\hline
\text{new } R_3: 0 \quad 7 \quad 2 \quad | \quad 4
\end{array}$$

let's assume  
you're not afraid  
of fractions for now.  
Can  $\rightarrow 7$   
 $\rightarrow 5$

$$\left[ \begin{array}{ccc|c} 3 & 2 & 1 & 8 \\ 0 & -35 & 7 & 14 \\ 0 & 35 & 10 & 20 \end{array} \right] \xrightarrow{R_3 + R_2 \rightarrow R_3}$$

$$\left[ \begin{array}{ccc|c} 3 & 2 & 1 & 8 \\ 0 & -35 & 7 & 14 \\ 0 & 0 & 17 & 34 \end{array} \right] \xleftarrow{\substack{\leftarrow \div 7 \\ \text{system OK} \\ \text{avoids fractions}}} \quad \text{Other books:}$$

Notice  $\frac{34}{5}$  is  
twice  $\frac{17}{5}$

I demand that  
you go all  
the way!

$$\left[ \begin{array}{ccc|c} 3 & 2 & 1 & 8 \\ 0 & -5 & 1 & 2 \\ 0 & 0 & 7 & 4 \end{array} \right] \xleftarrow{\substack{\text{want "1"} \\ \leftarrow \div 3 \text{ (maybe later)}}} \quad \xleftarrow{\leftarrow \div (-5)}$$

$$\left[ \begin{array}{ccc|c} 1 & \frac{2}{3} & \frac{1}{3} & \frac{8}{3} \\ 0 & 1 & -\frac{1}{5} & -\frac{2}{5} \\ 0 & 0 & 7 & 4 \end{array} \right] \xleftarrow{\substack{\text{want "0"} \\ \text{"in terms" }}} \quad \downarrow$$

$$\begin{array}{r} R_3: 0 \ 7 \ 2 \ 4 \\ (-7) \cdot R_2: 0 \ -7 \ \frac{2}{5} \ \frac{14}{5} \\ \hline \text{new } R_3: 0 \ 0 \ \frac{12}{5} \ \frac{34}{5} \end{array}$$

$$\left[ \begin{array}{ccc|c} 1 & \frac{2}{3} & \frac{1}{3} & \frac{8}{3} \\ 0 & 1 & -\frac{1}{5} & -\frac{2}{5} \\ 0 & 0 & \frac{12}{5} & \frac{34}{5} \end{array} \right] \xleftarrow{\leftarrow \div \left(\frac{12}{5}\right)} \quad \text{want "1"} \quad \text{(Some books allow you to stop reduction here. Key: Or below.)}$$

$$\left[ \begin{array}{ccc|c} 1 & x_2 & x_3 & \frac{8}{3} \\ 0 & \frac{2}{3} & \frac{1}{3} & -\frac{2}{5} \\ 0 & 1 & -\frac{1}{5} & \frac{34}{15} \end{array} \right]$$

→ System

$$\left\{ \begin{array}{l} x_1 + \frac{2}{3}x_2 + \frac{1}{3}x_3 = \frac{8}{3} \\ x_2 - \frac{1}{5}x_3 = -\frac{2}{5} \\ x_3 = 2 \end{array} \right.$$

Back-Sub

$$x_3 = 2$$

$$\begin{aligned} x_2 - \frac{1}{5}x_3 &= -\frac{2}{5} \\ x_2 - \frac{1}{5}(2) &= -\frac{2}{5} \\ x_2 - \frac{2}{5} &= -\frac{2}{5} \\ x_2 &= 0 \end{aligned}$$

$$\begin{aligned} x_1 + \frac{2}{3}x_2 + \frac{1}{3}x_3 &= \frac{8}{3} \\ x_1 + \frac{2}{3}(0) + \frac{1}{3}(2) &= \frac{8}{3} \\ x_1 + \frac{2}{3} &= \frac{8}{3} \\ x_1 &= \frac{6}{3} \\ x_1 &= 2 \end{aligned}$$

$$\{(2, 0, 2)\}$$

Can ✓

Up to  
25,  
except  
7, 9, 11, 21

See (Ex 3) (p. 16), (Ex 5) (p. 18) - for more practice.

Watch order  
here - lucky -  
palindrome

E) When Does a System Have No Solution?

If we ever get a row

$$\begin{array}{cccc|c} 0 & 0 & \dots & 0 & | \text{ a non-0 #} \\ \underbrace{\quad}_{\substack{\text{(all "0"s on)} \\ \text{the coeff. side}}} & & & & \underbrace{\quad}_{\text{RHS}} \end{array}$$

**STOP!** Sol'n set =  $\emptyset$   
System "inconsistent"

Ex  $\begin{cases} x+y=1 \\ x+y=4 \end{cases}$

$$\left[ \begin{array}{cc|c} 1 & 1 & 1 \\ 1 & 1 & 4 \end{array} \right]$$

$$R_2 + (-1)R_1$$

$$R_2 - R_1 \rightarrow R_2$$

$$\left[ \begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 0 & 3 \end{array} \right] \leftarrow 0=3 \text{ can never be satisfied}$$

$\emptyset$

See Ex 6 (p. 20)

Up to 25  
except  
7, 9, 11

## (F) Row-Echelon Form for a Matrix

We aimed for

$$\left[ \begin{array}{cccc|c} 1 & & & & \\ \cdot & 1 & & & \\ 0 & \cdot & 1 & & \\ \vdots & & & \ddots & \\ 0 & \cdot & \cdot & \cdot & 1 \end{array} \right] \quad \text{special care of}$$

What if we can't get this? Ex [square]

isn't even square?)

In general, we aim for row-echelon form.

### Properties

① If there are any "all-0" rows, they must be at the bottom.

Aside from "all-0" rows,

② Every row must have a "leading 1" as its leftmost non-0 entry.

③ The "leading 1" of a row must always be to the right of the "leading 1's" of all higher rows. ( $\downarrow \rightarrow L_1, L_2, \dots$ )

They go down and to the right. JFC

③ The "leading 1's" go down and to the right. (JFC)

1.2.14

Is this matrix  
in row echelon  
form?

Ex  $\left[ \begin{array}{cccc|c} 0 & 1 & 3 & 1 & 4 & 7 \\ 0 & 0 & 0 & 1 & 9 & 4 \\ 0 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right]$

("Leading 1's are in the pivot positions)

Ex Solve  $\begin{cases} 2x - 3y = 21 \\ 4x - 6y = 42 \\ 2x - 2y = 16 \end{cases}$

$R_2 - 2R_1 \rightarrow$  or  $\left[ \begin{array}{cc|c} 2 & -3 & 21 \\ 0 & -6 & 42 \\ 2x - 2y = 16 \end{array} \right]$ .

What do  
we get?  
Walk around.

or  $\div 2$  1st

$\left[ \begin{array}{cc|c} 2 & -3 & 21 \\ 0 & 0 & 0 \\ 0 & 1 & -5 \end{array} \right] \xrightarrow{\text{S}}$

$\left[ \begin{array}{cc|c} 2 & -3 & 21 \\ 0 & 1 & -5 \\ 0 & 0 & 0 \end{array} \right] \xleftarrow{\div 2}$

$\left[ \begin{array}{cc|c} 1 & -\frac{3}{2} & \frac{21}{2} \\ 0 & 1 & -5 \\ 0 & 0 & 0 \end{array} \right]$

1.2.15

→ System

$$\begin{cases} x - \frac{3}{2}y = \frac{21}{2} \\ y = -5 \\ 0 = 0 \end{cases} \quad \left( \begin{array}{l} \text{any sol'n of this} \\ \text{automatically satisfies "0=0"} \end{array} \right)$$

← Can drop

Back-Sub

$$(y = -5)$$

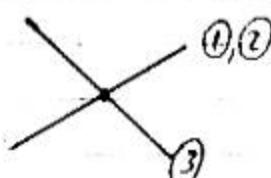
$$\begin{aligned} x - \frac{3}{2}y &= \frac{21}{2} \\ x - \frac{3}{2}(-5) &= \frac{21}{2} \\ x + \frac{15}{2} &= \frac{21}{2} \\ (x = \frac{3}{2}) \\ x &= 3 \end{aligned}$$

$$\{(3, -5)\}$$

Can ✓

Note

$$\begin{cases} 2x - 3y = 21 \\ 4x - 6y = 42 \\ 2x - 2y = 16 \end{cases} \quad \left\{ \begin{array}{l} \text{same stupid} \\ \text{line!} \end{array} \right.$$



## ⑥ Reduced Row-Echelon (RRE) Form

Recite ①-③

0's directly  
above/below

Row-echelon form and

Property ④: Each leading 1 has all "0's elsewhere in its column.

Old Ex Solve  $\begin{cases} 2x - 3y = 21 \\ 4x - 6y = 42 \\ 2x - 2y = 16 \end{cases}$

using Gauss-Jordan elimination  
( $\rightarrow$  RRE form).

Augmented Matrix

$$\left[ \begin{array}{cc|c} 2 & -3 & 21 \\ 4 & -6 & 42 \\ 2 & -2 & 16 \end{array} \right]$$

Which # must  
 $\rightarrow 0$   
We need to  
eliminate up  
from the  
leading 1's.

Old Ex  $\left[ \begin{array}{cc|c} 1 & -\frac{3}{2} & \frac{21}{2} \\ 0 & 1 & -5 \\ 0 & 0 & 0 \end{array} \right]$  Need "0" for RRE form  
Row-Echelon Form

In Gauss-Jordan elimination, we  $\rightarrow$  RRE form

$$\begin{array}{rcl} R_1: 1 & -\frac{3}{2} & \frac{21}{2} \\ +\frac{3}{2}R_2: 0 & \frac{3}{2} & -\frac{15}{2} \\ \hline \text{new } R_1: 1 & 0 & 3 \end{array} \quad \left. \begin{array}{l} \text{"Work" moves from} \\ \text{Back-Sub stage} \\ \text{to here} \end{array} \right\}$$

1.2.17

$$\left[ \begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & -5 \\ 0 & 0 & 0 \end{array} \right] \quad \underline{\text{RREF form}}$$

$$\begin{cases} x = 3 \\ y = -5 \end{cases} \quad (\text{Sol'n pops out!})$$

MTH 146  
If you're working  
with friends

Note Every matrix has a unique RREF form.

Ex 7 (p. 21)

Book:

Row-echelon form  $\rightarrow$  RREF form

Book:  $4x + 0$

$$\left[ \begin{array}{ccc|c} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 2 \end{array} \right]$$

Faster: "Clear up" columns from  
right-to-left.

(You get to take advantage  
of "0's along the way.)

We like to take  
advantage of D.  
He's easy to  
push around.  
Take his lunch  
money.

## (H) Systems with $\infty$ Many Sol'n's

Col of 0s  
see Lf-4+

Ex Solve  $\begin{cases} x_1 - 2x_2 + x_3 + 5x_4 = 3 \\ 2x_1 - 4x_2 + x_3 + 7x_4 = 5 \end{cases}$

Augmented Matrix

$$R_2 - 2R_1 \rightarrow \left[ \begin{array}{ccc|cc} 1 & -2 & | & 5 & 3 \\ 0 & -4 & | & 7 & 5 \end{array} \right]$$

$$\cdot(-1) \rightarrow \left[ \begin{array}{ccc|cc} 1 & -2 & | & 5 & 3 \\ 0 & 0 & | & -3 & -1 \end{array} \right]$$

$$R_1 - R_2 \rightarrow \left[ \begin{array}{ccc|cc} 1 & -2 & | & 5 & 3 \\ 0 & 0 & | & 1 & 1 \end{array} \right] \quad \text{Row-Echelon Form}$$



Free vars. correspond to columns in the coeff. matrix that do not have a leading 1.  
(Here,  $x_2$  and  $x_4$ .)

If there are free vars and the system is consistent (no row of 0's  $\neq 0$ ),

You must  $\rightarrow$  RRE form to describe the sol'n set.

1.2.19

RRE Form

$$\left[ \begin{array}{cccc|c} 1 & -2 & 0 & 2 & 2 \\ 0 & 0 & 1 & 3 & 1 \end{array} \right] .$$

$x_1$        $x_2$        $x_3$        $x_4$   
 (free)      (free)

→ System

$$\begin{cases} x_1 - 2x_2 + 2x_4 = 2 \\ x_3 + 3x_4 = 1 \end{cases}$$

Move all free vars. to the right side.

$$\begin{cases} x_1 = 2 + 2x_2 - 2x_4 \\ x_3 = 1 - 3x_4 \end{cases}$$

Parametrization

Let  $x_2 = t$  (g book uses)  
 $x_4 = u$  (s, t)

Sol'n Set in Parametric Form

$$\begin{aligned} x_1 &= 2 + 2t - 2u \\ x_2 &= t \\ x_3 &= 1 - 3u \\ x_4 &= u \end{aligned}$$

$t, u$  are any real #s.

1.2.20

Some Sol'n's

$$\left[ \begin{array}{cc|cccc} 1 & 4 & x_1 & x_2 & x_3 & x_4 \\ 0 & 0 & 2 & 0 & 1 & 0 \\ -4 & 7 & -20 & -4 & -20 & 7 \\ \vdots & & & & & \end{array} \right]$$

2D plane of sol'n's in 4D space.  
2 free vars                          4 vars

See Ex 8 (p. 22)

## I) Homogeneous Systems of Linear Eqs

(have augmented matrices that have)  
all "0"s in RHS.

$$\left[ \begin{array}{c|c} & \text{RHS} \\ \hline & \begin{matrix} 0 \\ 0 \\ \vdots \\ 0 \end{matrix} \end{array} \right]$$

Ex  $\begin{cases} x + y - z = 0 \\ 2x + y - 3z = 0 \end{cases}$  Consistent

We (automatically) know  $(\overset{x}{0}, \overset{y}{0}, \overset{z}{0})$   
is a sol'n (the trivial sol'n).

(Any such system is consistent.)

If more vars. than eqs., then

$$\left[ \begin{array}{c|c} \text{"fat"} & | 0 \\ \text{(coeff.)} & | \vdots \\ \text{(matrix)} & | 0 \end{array} \right]$$

we're going  
to have what  
kind of vars?

There will be free vars.

$\infty$  many sol'n's.

1.2.22

Note

$$\begin{matrix} \text{square} & \left[ \begin{array}{c|c} 0 \\ \vdots \\ 0 \end{array} \right] & \text{slimy} & \left[ \begin{array}{c|c} 0 \\ \vdots \\ 0 \end{array} \right] \end{matrix}$$

can have 1 or  $\infty$  many sol'n's.Ex (You)Homog. system w/more eqs. than vars.  
that has  $\infty$  many sol'n'sWrite  $\Rightarrow$   
They got it!

$$\left\{ \begin{array}{l} x + y = 0 \\ 2x + 2y = 0 \\ 3x + 3y = 0 \end{array} \right. \quad \begin{array}{l} \text{same stupid} \\ \text{can drop} \\ \text{line} \end{array}$$

$$\left[ \begin{array}{cc|c} 1 & 1 & 0 \\ 2 & 2 & 0 \\ 3 & 3 & 0 \end{array} \right]$$

$$\left[ \begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right]$$

$x$        $y$   
(free)

Too early

Ex 1 sol'n

$$\left\{ \begin{array}{l} x = 0 \\ y = 0 \\ 2y = 0 \end{array} \right. \rightarrow \left[ \begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right]$$

$$\{(0,0)\}$$

# ① Combos

Go col by col

square | ]

	# sol's
0	1

1

∞ many

$$\left[ \begin{array}{c|c} 1 & 2 \\ 0 & 0 \end{array} \right]$$

$$\left[ \begin{array}{c|c} 1 & 0 \\ 0 & 1 \end{array} \right]$$

$$\left[ \begin{array}{c|c} 1 & 1 \\ 0 & 0 \end{array} \right]$$

A skinny can  
do whatever  
a square can  
do

skinny | ]

$$\left[ \begin{array}{c|c} 1 & 2 \\ 0 & 0 \\ 0 & 0 \end{array} \right]$$

$$\left[ \begin{array}{c|c} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{array} \right]$$

$$\left[ \begin{array}{c|c} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{array} \right]$$

fat | ]

$$\left[ \begin{array}{c|c} 1 & 2 \\ 0 & 0 \\ 0 & 0 \end{array} \right]$$

(NO!)

$$\left[ \begin{array}{c|c} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{array} \right]$$

(Is there a relationship between matrix size  
and # sols.? Not really, except (NO!).  
(Other than that, anything goes!))

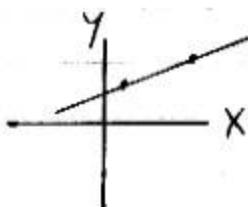
## 1.3: APPLICATIONS

### A) Polynomial Curve Fitting in $\mathbb{F}_x$

Why diff?  
graph can't fail  
VLT

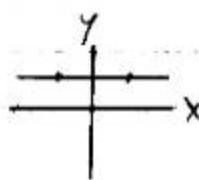
n points w/different x-coords.  
determine a polynomial function  
of degree  $\leq n-1$ .

2 pts.



$$p(x) = a_0 + a_1 x \quad (\text{deg. 1})$$

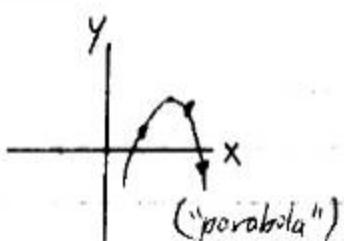
Special Case



$$p(x) = a_0 \quad (\text{deg. 0})$$

Here,  $a_1 = 0$ .

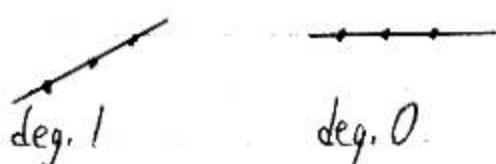
3 pts.



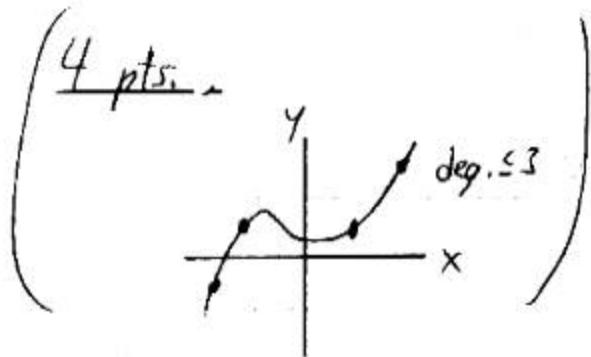
$$p(x) = a_0 + a_1 x + a_2 x^2 \quad (\text{deg. 2})$$

("parabola")

Special curves



There are higher deg. funcs. that will do this..



n pts.

$p(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_{n-1} x^{n-1}$  will do

" $a_i$ 's can be 0

$\deg \leq n-1$

Ex Find a polynomial function where graph passes through  $(-1, 2)$ ,  $(0, -1)$ , and  $(3, 2)$ .

$\underbrace{p(x)}_{\text{or } y} = a_0 + a_1 x + a_2 x^2$  will do

Plug in

$$\begin{aligned} (-1, 2) \quad 2 &= a_0 + a_1(-1) + a_2(-1)^2 \\ (0, -1) \quad -1 &= a_0 + a_1(0) + a_2(0)^2 \\ (3, 2) \quad 2 &= a_0 + a_1(3) + a_2(3)^2 \end{aligned}$$

In practice  
plug  $a_0 = -1$   
into other  
systems of  
eqns.

Solve  $\begin{cases} a_0 - a_1 + a_2 = 2 \\ a_0 = -1 \\ a_0 + 3a_1 + 9a_2 = 2 \end{cases}$

use (Gauss + etc)  
matrix  
constant  
- is always  
 $x_1, x_2, \dots, x_n$   
basis for

What's the col?  
mt 20 Num.  
Recipes

Vandermonde matrix

$$\begin{bmatrix} 1 & -1 & 1 & 2 \\ 1 & 0 & 0 & -1 \\ 1 & 3 & 9 & 2 \end{bmatrix}$$

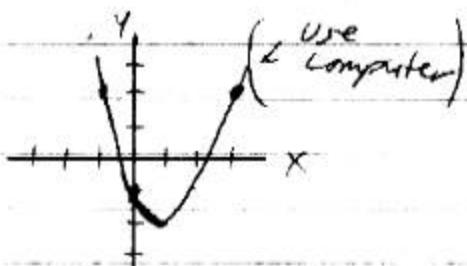
"1"s "x"s "x^2"s "y"s (Shortcut (4 pt.  $\rightarrow$  cubic  $\rightarrow$   $x^3$ ))

$$\text{Solution: } \begin{cases} a_0 = -1 \\ a_1 = -2 \\ a_2 = 1 \end{cases}$$

$$p(x) = a_0 + a_1 x + a_2 x^2$$

$$p(x) = -1 - 2x + x^2$$

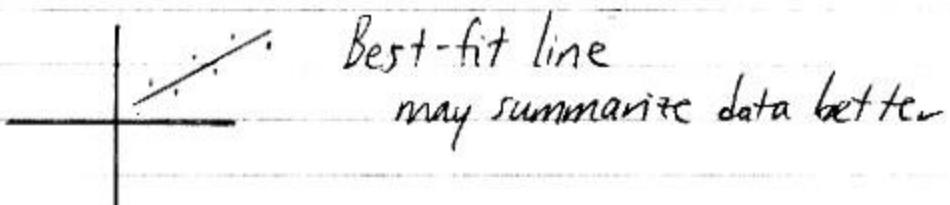
$$x^2 - 2x - 1$$



Did you find a  
quadratic?

You don't want  
your model to  
be too dependent on  
noise

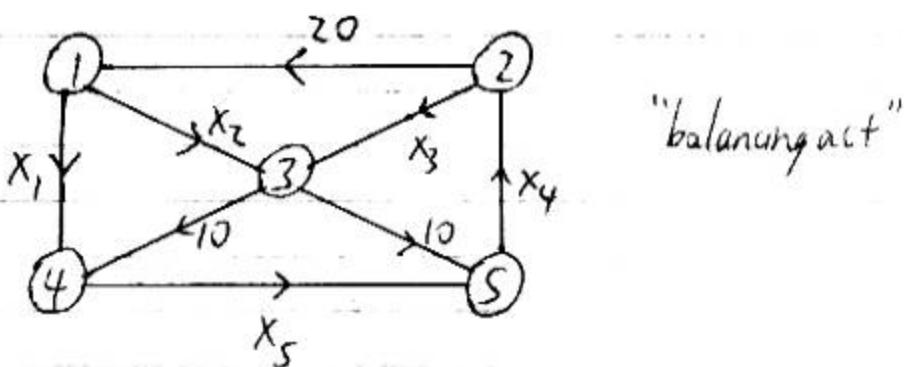
In stats



## B) Network Analysis

Read Ex. 5 on pp. 31-32

5 junctions (circles)



(At each junction,) flow in  $\rightarrow$   $\text{O} \rightarrow$  flow out

$$\begin{array}{ccc} \textcircled{1} & \xleftarrow{\text{20 (in)}} & \\ \downarrow & & \\ x_1 & \xrightarrow{\text{x2 (out)}} & \\ & \text{(out)} & \end{array} \quad 20 = x_1 + x_2$$

$$\left. \begin{array}{ccc} \textcircled{2} & \xleftarrow{\text{20 (out)}} & \\ \downarrow & & \\ x_3 & \xrightarrow{\text{x4 (in)}} & \\ & \text{(out)} & \end{array} \right\} \quad x_4 = x_3 + 20$$

etc.

We get a system of 5 linear eqs. Maybe  $\infty$  solns.

May require all  $x_i \geq 0$ , integer.