

CH 2: MATRICES2.1: MATRIX OPERATIONS① Notation A, B, C, \dots usually denote matrices.

$$A = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & \dots \\ & \ddots & \\ & & a_{mn} \end{bmatrix}$$

(i.e., A consists of entries " a_{ij} " $A = [a_{11} \ a_{12} \ \dots \ a_{mn}]$)

$$B = [b_{ij}]$$

row vector: (a matrix w/only 1 row.)

Ex $\begin{bmatrix} 1 & 0 & \sqrt{2} \end{bmatrix}$
 components

column vector: (a matrix w/only 1 column.)

- Denoted by \vec{a}, \vec{b}, \dots
 boldfaced

(Ex $\vec{a} = \begin{bmatrix} 5 \\ -1 \\ 3/2 \end{bmatrix}$)

$$A = [\vec{a}_1 \ \vec{a}_2 \ \dots \ \vec{a}_n]$$

i.e., A can be partitioned
 into n column vectors

⑧ $A = B$

$\Leftrightarrow \left\{ \begin{array}{l} \text{if and only if} \\ \text{① same size, and} \\ \text{② corresponding entries are } = \end{array} \right.$

$$\text{Exs } [1 \ 0] = [1 \ 0]$$

As vectors,
maybe =.

$$[1 \ 0] \neq \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

(1x2) (2x1)

$$[1 \ 0] \neq [0 \ 1]$$

⑨ $A + B$

To obtain $A + B$, add corresp. entries.

$$\text{Ex } \underbrace{\begin{bmatrix} 1 & -2 \\ 3 & 0 \end{bmatrix}}_A + \underbrace{\begin{bmatrix} -1 & -3 \\ 1 & 2 \end{bmatrix}}_B \stackrel{\text{add}}{=} \underbrace{\begin{bmatrix} 0 & -5 \\ 4 & 2 \end{bmatrix}}_{A+B}$$

If diff. sizes, $A + B$ is undefined.
 $(A, B$ must have the same size, otherwise (o.w.)
 $\text{"+ compatible"})$

$A - B$: subtract corresp. entries

D) cA (Scalar Multiplication)

↙
a real #
(scalar)

To obtain cA , multiply each entry of A by c .

Ex $\underbrace{5 \begin{bmatrix} 2 & -\frac{3}{5} \\ 0 & 1 \end{bmatrix}}_{c \ A} = \begin{bmatrix} 10 & -3 \\ 0 & 5 \end{bmatrix}$

Can do 1, 3

E) (Row vector) times (Column vector)

$$[a_1 \ a_2 \ \dots \ a_n] \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = a_1 b_1 + a_2 b_2 + \dots + a_n b_n$$

Add products of corresp. entries.
Think: Dot product

Ex $\begin{bmatrix} 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \\ -1 \end{bmatrix}$

$$= (1)(4) + (0)(5) + (3)(-1)$$

$$= 4 + 0 - 3$$

$$= 1 \quad \text{or } [1]$$

(scalar or maybe)
(x/matrix. [1])

Take the
product of the
1st entries...

Booker sloppy

if you treat
them as matrices.

Can do 1, 3

(F) AB (Matrix Multiplication)

$$AB = "C"$$

$$= [c_{ij}]$$

where $c_{ij} = (\text{i}^{\text{th}} \text{ row of } A) \text{ times}$
 $(\text{j}^{\text{th}} \text{ column of } B)$

Seems
weird
componentwise
multi. is
diff.
6.3 ⑩ - we'll
see why.

Ex

$$\begin{bmatrix} 1 & -1 & 0 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} 4 & -1 \\ 3 & 0 \\ 1 & 2 \end{bmatrix}$$

$$\begin{array}{c} A \\ 2 \times 3 \end{array} \quad \begin{array}{c} B \\ 3 \times 2 \end{array}$$

need = (Multi. compatible),
o.w., AB
is undefined

Trick

$$\begin{bmatrix} 4 & -1 & 0 \\ 3 & 0 & 1 \\ 1 & 2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 0 \\ 2 & 1 & 3 \end{bmatrix} \xrightarrow{c_{11} - c_{12}} \begin{bmatrix} 1 & -1 & 0 \\ 2 & 1 & 3 \end{bmatrix} \xrightarrow{c_{21} - c_{22}} \begin{bmatrix} 1 & -1 & 0 \\ 2 & 1 & 3 \end{bmatrix} \xrightarrow{AB}$$

(You)

$$\begin{aligned} c_{11} &= (1)(4) + (-1)(3) + (0)(1) = 1 \\ c_{12} &= -1 + 0 + 0 = -1 \\ c_{21} &= 8 + 3 + 3 = 14 \\ c_{22} &= -2 + 0 + 6 = 4 \end{aligned}$$

some have seen

$$AB = \begin{bmatrix} 1 & -1 \\ 14 & 4 \end{bmatrix}$$

$\underbrace{\begin{array}{c} A \\ m \times q \end{array}}_{m \times p} \quad \underbrace{\begin{array}{c} B \\ n \times p \end{array}}_{n \times q}$
 $\begin{pmatrix} [A] & [AB] \end{pmatrix}$

what do you
think the size
of the product
is?

$n \times p$ and
 $m \times n$

Often, $AB \neq BA$ (In fact, if $m \neq p$, BA is undefined.)

Matrix mult. is not commutative. ("+" is)

why are
we doing
this way?
all but 19
A^{1/3}: decompose A^{1st}

Read Ex on pp. 45-46 (Applc.)
 $A^k = \underbrace{AA \cdots A}_{k \text{ copies}}$ (if A is square, k is a natural #)

⑥ Matrix Notation and Systems of Linear Eqs

Form $A\vec{x} = \vec{b}$
where the
augmented matrix
is $[A | \vec{b}]$
(coeff.) (RHS)
(matrix)

Ex. 8 (1,2)

$$\text{Ex } \begin{cases} 2x_1 + 4x_2 - 2x_3 = 0 \\ 3x_1 + 5x_2 = 1 \end{cases}$$

Augmented matrix

$$\left[\begin{array}{ccc|c} 2 & 4 & -2 & 0 \\ 3 & 5 & 0 & 1 \end{array} \right]$$

$\underbrace{\quad}_{A} \qquad \underbrace{\quad}_{\vec{b}}$

So, the system can be written as

$$\left[\begin{array}{ccc} 2 & 4 & -2 \\ 3 & 5 & 0 \end{array} \right] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$A \quad \vec{x} = \vec{b}$$

(Why?

$$\left(\begin{array}{ccc|c} 2 & 4 & -2 & 0 \\ 3 & 5 & 0 & 1 \end{array} \right) \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_1 + 4x_2 - 2x_3 \\ 3x_1 + 5x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Solve $[A : \vec{b}] \Rightarrow$

$$\begin{cases} x_1 = 2 - 5t \\ x_2 = -1 + 3t \\ x_3 = t \end{cases} \quad t \text{ is any real #}$$

seems
backward

$$\begin{cases} x_1 = 2 + t(-5) \\ x_2 = -1 + t(3) \\ x_3 = 0 + t(1) \end{cases}$$

Vector notation

$$\underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}}_{\text{"Initial pt."}} + t \underbrace{\begin{bmatrix} -5 \\ 3 \\ 1 \end{bmatrix}}_{\text{"Direction vector"}}$$

Discuss later.

If I graph the
sol'n set in
3-space,
what do I
get?

oo many sol'n's

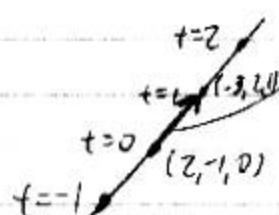
$$t=0 \rightarrow \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$$

$$t=1 \rightarrow \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}$$

$$t=2 \rightarrow \begin{bmatrix} -8 \\ 5 \\ 2 \end{bmatrix}$$

etc.

$$\begin{bmatrix} -5 \\ 3 \\ 1 \end{bmatrix}$$



2.2: ALGEBRA OF MATRICES

Read (skip proof on p. 61)

① Matrix Eqs.

meaning that
they're the same
size

Ex Let A, B, X be $m \times n$

Solve $B + 4X = 7A$ for X

Can "+" or "-" same matrix
on both sides.

$$4X = 7A - B$$

" \div by 4" awkward (for
matrices)

Let's "mult. by $\frac{1}{4}$ ".

$$\frac{1}{4}(4X) = \frac{1}{4}(7A - B)$$

$$X = \frac{1}{4}(7A - B) \quad \leftarrow \text{easier to compute}$$

or (can distribute $\frac{1}{4}$)

$$X = \frac{7}{4}A - \frac{1}{4}B$$

③ Matrix Multiplication) Properties

So that all
additions, mults.
are defined.

Assume size-compatibility.

① Associative

$$(AB)C = A(BC)$$

② Not commutative

$$AB \text{ may or may not} = BA$$

③ Distributive props.

$$\begin{aligned} A(B+C) &= AB+AC \\ (A+B)C &= AC+BC \end{aligned}$$

Both true
only possible
if A, B, C -n.n
if sizes compatible, mult. on the right by C

④ If $A=B$, then $AC=BC$ (if sizes compatible)

$$CA=CB$$

left

⑤ But can't always cancel.

$$\text{Algebra I: } ac=bc$$

$$\Rightarrow a=b \text{ (provided } c \neq 0)$$

If $AC=BC$,
 A may or may not = B

There are many
s.t., where
 $AC=BC$ but $A \neq B$.

not just the
matrix

$$\text{Ex } \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 5 \\ 9 & 9 \end{bmatrix}$$

A C AC

$$\begin{bmatrix} 1 & 4 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 5 \\ 9 & 9 \end{bmatrix}$$

B C BC

$$AC = BC, \text{ but } A \neq B.$$

\leftarrow A counterexample to the claim " $AC = BC \rightarrow A = B$ "

a "false" stmt, even though sometimes true.

True means
always true
False means
sometimes false
(given the
assumptions)

(C) Special Matrices

A zero matrix "0" has all entries = 0.

$$\text{Ex } \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad "0_{2 \times 3}"$$

An identity matrix "I" is a square matrix with:

"1"s along the
main diagonal

\downarrow

$$\begin{bmatrix} 1 & & & \\ & 0 & & \\ & & 0 & \\ & & & 1 \end{bmatrix} \quad "0"s \text{ elsewhere}$$

I_n = the $n \times n$ identity

$$\text{Ex } I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

If A is $m \times n$,

$$\underset{(m \times m)}{A} \underset{(n \times n)}{I_n} = A$$

$$\underset{(m \times m)}{I_m} \underset{(n \times n)}{A} = A$$

Ex

$$\begin{bmatrix} 4 & 5 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 5 \end{bmatrix}$$

I_2

length of
row
= length of
col
 $[4 \ 5] [] = [4 \ 5]$

① A^T (A Transpose)

If A is $m \times n$.

Interchange rows \leftrightarrow columns to obtain A^T .

$(n \times m)$

$$\text{Ex If } A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \quad (2 \times 3)$$

$$\text{then } A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} \quad (3 \times 2)$$

Properties

(Assume sizes compatible)

$$\textcircled{1} \quad (A^T)^T = A \quad (\text{Key})$$

It doesn't matter if you add 1st and then flip or..

$$\textcircled{2} \quad (A+B)^T = A^T + B^T$$

It doesn't matter if you 'c' 1st or flip 1st.

$$\textcircled{3} \quad (cA)^T = c(A^T)$$

$$\textcircled{4} \quad (AB)^T = B^T A^T \quad (\text{Key})$$

Transpose of a product =
Reverse product of transposes

E) Symmetric MatricesSquare matrix "A" is Symmetric $\Leftrightarrow A = A^T$

Ex

$$\begin{bmatrix} 1 & 0 & 4 \\ 0 & 3 & 5 \\ 4 & 5 & -1 \end{bmatrix}$$

$\underbrace{\hspace{1cm}}_{\text{mirror}}$

rows same as cols.

= what?

$$a_{ij} = a_{ji} \quad \text{for all } 1 \leq i \leq n, \\ 1 \leq j \leq n$$

What does A^{-1}
mean? 6.3 @

2.3: A^{-1} (A Inverse)

④ Definition

$$3 \cdot \frac{1}{3} = 1$$

↑ mult. inverses ↓ mult. id.

Assume A, B, C, \dots are square $n \times n$ matrices.

What are the
analogies in
matrix algebra?

A has either no inverses or exactly one (A^{-1}).

\downarrow

A is noninvertible A is invertible
or singular or nonsingular.

If A is invertible, there is a unique $n \times n$ matrix A^{-1} that satisfies:

$$AA^{-1} = I_n \quad \text{if one holds} \Rightarrow$$

and $A^{-1}A = I_n \quad \text{if both hold.}$

To show $B = A^{-1}$, just show $AB = I_n$.

Then, $BA = I_n$ is automatic.

Note: In other books, nonsquare matrices may have left, right inverses.

NO!!

IGNORE

Ch 4.21

$AB = I$

A noninv. \rightarrow

$A \neq I$

A nonsquare

row

$E_A \rightarrow E_A B$

which is

inv. b/c

a P row

\vdash

$\therefore A$ m/s

Hoffman p.24

$BA = I$

$\rightarrow AX = 0$ has

only 0 soln

$\therefore A^{-1}$

$A^{-1}X = 0$

Proof (p. 67) - uniqueness

B - right inverse of A
 C - left

$$AB = I$$

$$\underbrace{CAB}_I = CI$$

$$B = C$$

Book: 2-sided inv.
2-sided inv. \Rightarrow same,

right inv. \Rightarrow same ; A^{-1} is unique

③ Finding A^{-1}

Ex If $A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$, find A^{-1} .

$$\text{i.e., } \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} [\ ? \] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

see 2.1 #25

$\approx \frac{1}{5}$
understood

Idea

$$\begin{bmatrix} A \end{bmatrix} \left[\begin{array}{c|c} \vec{x}_1 & \vec{x}_2 \end{array} \right] \xrightarrow{A^{-1}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Good idea to
bring up
Keep "idea"

See 2.1 #53 (Block multiplication)

Simultaneously solve 2 systems:

$$A\vec{x}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$A\vec{x}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Same Coeff. matrix Solving for cols of A^{-1}

Makes things very convenient for us.

Method 1 (Gauss-Jordan Elim.)

$$\text{Ex } \left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 2 & 4 & 0 & 1 \end{array} \right] \quad \text{"adjoining"}$$

Set up $[A | I]$
 $\downarrow \text{EROS}$
 $[I | A^{-1}]$

$$\begin{array}{l} R_2 : 2 \quad 4 \quad | \quad 0 \quad 1 \\ +(-2)R_1 : -2 \quad -6 \quad | \quad -2 \quad 0 \\ \hline \text{new } R_2 : 0 \quad -2 \quad | \quad -2 \quad 1 \end{array}$$

$$\left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & -2 & -2 & 1 \end{array} \right] \leftarrow \div (-2)$$

$$\left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & 1 & 1 & -\frac{1}{2} \end{array} \right]$$

$$R_1 + (-3)R_2 \rightarrow R_1$$

$$\left[\begin{array}{cc|cc} 1 & 0 & -2 & \frac{3}{2} \\ 0 & 1 & 1 & -\frac{1}{2} \end{array} \right] \xrightarrow{\text{A}^{-1}}$$

$\underbrace{I}_{A \vec{x}_1 = [1]} \quad \vec{x}_1 \quad \vec{x}_2 \quad A \vec{x}_2 = [0]$

In general ($n \times n$)

$$\begin{array}{c} [A \mid I_n] \\ \downarrow \text{EROS} \\ [I_n \mid A^{-1}] \end{array}$$

\vec{x}_1 becomes

If you ever get a row of "0's" in the left square, you can't get $[I \mid A^{-1}]$, and A^{-1} doesn't exist. (A is singular).

$$\text{Ex } [0 \ 0 \ 0 \mid 3 \ 5 \ 4]$$

Method 2 (Shortcut Formula Only for 2×2 Matrices)

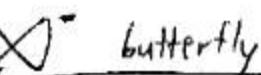
$$\text{If } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

then the determinant of A

not abs. value

$$= \det(A), \text{ or } |A|$$

$$= ad - bc$$

Think:  butterfly

skew diagonal

and
$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

adjoint of A switch signs

If $\det(A) = 0$, A^{-1} doesn't exist.

old

$$\text{Ex } A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

$$\begin{aligned} \det(A) &= (1)(4) - (3)(2) \\ &= -2 \end{aligned}$$

$$A^{-1} = \frac{1}{-2} \begin{bmatrix} 4 & -3 \\ -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & \frac{3}{2} \\ 1 & -\frac{1}{2} \end{bmatrix}$$

Know both methods!

C) Properties

Assume for now: inverses exist,
sizes compatible

(Key)

$$\textcircled{1} \quad (A^{-1})^{-1} = A \quad \left. \begin{matrix} \text{like } A^T \\ \text{props.} \end{matrix} \right\}$$

$$\textcircled{2} \quad (AB)^{-1} = B^{-1}A^{-1} \quad \left. \begin{matrix} \text{like } A^T \\ \text{props.} \end{matrix} \right\}$$

Proof Show: the inverse of AB is $B^{-1}A^{-1}$.
Sufficient to show $(AB)(B^{-1}A^{-1}) = I$.

$$\begin{aligned} & (AB)(B^{-1}A^{-1}) \\ &= A \underbrace{(BB^{-1})}_{I} A^{-1} \quad (\text{Matrix mult. is assoc.}) \\ &= \underbrace{AA^{-1}}_{I} \\ &= I \end{aligned}$$

Remember
series
 $1-3+3-5+5-7$

"telescoping proof"



cancellations

Is this B?
Not nec.

Warning ABA^{-1} (Matrix mult. is not comm.)
"in the way"

③ Cancellation Props.

If C is invertible,

$$\begin{array}{l} AC = BC \rightarrow A = B \\ CA = CB \rightarrow A = B \end{array} \quad \text{(HW)}$$

Proof

$$\left. \begin{array}{l} AC = BC \\ ACC^{-1} = BCC^{-1} \\ A = B \end{array} \right\} \quad \left. \begin{array}{l} CA = CB \\ C^{-1}CA = C^{-1}CB \\ A = B \end{array} \right\}$$

(Also)

can't invert
0

$$④ (cA)^{-1} = \frac{1}{c} A^{-1}, \text{ if } c \neq 0$$

$$\text{Ex } (3A)^{-1} = \frac{1}{3} A^{-1}$$

$$\checkmark: (3A)(\frac{1}{3}A^{-1}) = \underbrace{(3)}_{I} \underbrace{(\frac{1}{3})}_{I} \underbrace{(AA^{-1})}_{I} = I$$

$$⑤ (A^T)^{-1} = (A^{-1})^T$$

The inverse and transpose ops. commute.

$$⑥ \underbrace{(A^k)^{-1}}_{\text{call } A^{-k}} = \underbrace{(A^{-1})^k}, \text{ if } k \text{ is a natural \#}$$

"Inverse" and exponentiation commute.

T of a sum
= sum of T 's

Warning

$$(A+B)^T = A^T + B^T, \text{ but } (A+B)^{-1} \neq A^{-1} + B^{-1}, \text{ usually}$$

① Inverses and Systems

System $A\vec{x} = \vec{b}$ where A is square.

If A is invertible \rightarrow

This system has 1 unique sol'n,
namely $(\vec{x} = A^{-1}\vec{b})$.

Me is why?
suff. for
uniqueness
it we assume
 A^{-1} is unique?
 $\vec{x} = A^{-1}\vec{b}$
if $\vec{x} + A^{-1}\vec{b}$
 $\vec{x} \neq A^{-1}\vec{b}$

Uniqueness pt
doesn't require
uniqueness of
 A^{-1} .

$$\begin{aligned} \text{Why? } A\vec{x} &= \vec{b} \\ \Rightarrow \underbrace{A^{-1}A\vec{x}}_{I} &= A^{-1}\vec{b} \\ \Rightarrow \vec{x} &= A^{-1}\vec{b} \end{aligned}$$

$$\left(\begin{array}{l} \text{Why unique?} \\ \text{If } A\vec{x}_1 = \vec{b} \\ A\vec{x}_2 = \vec{b} \\ \Rightarrow A\vec{x}_1 = A\vec{x}_2 \\ \Rightarrow \vec{x}_1 = \vec{x}_2 \end{array} \right)$$

If A is singular \rightarrow

This system has either no sol'n
or ∞ many.

Ex Solve $\begin{cases} 2x_1 + 5x_2 = 4 \\ x_1 + 3x_2 = 3 \end{cases}$

" A is sing.
I could have
 $\vec{b} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$
1 sol'n.

$$\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

Optional

$$\begin{aligned} \det(A) &= (2)(3) - (5)(1) \\ &= 1 \leftarrow \text{not } 0, \text{ so } A \text{ is invertible} \end{aligned}$$

Find A^{-1} (easy for 2×2)

$$\frac{1}{\det(A)} = 1$$

$$A^{-1} = \frac{1}{1} \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$$

$$\vec{x} = A^{-1} \vec{b}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} -3 \\ 2 \end{bmatrix}$$

$$\begin{cases} x_1 = -3 \\ x_2 = 2 \end{cases}$$

Criticisms of $\vec{x} = A^{-1} \vec{b}$ Method

Only works when A square, invertible.

Finding $A^{-1} \vec{b}$ is less efficient for solving larger systems, (compared to Ch. 1 methods,) unless you want to solve several systems w/same A :

$$\begin{aligned} A \vec{x}_1 &= \vec{b}_1 \\ A \vec{x}_2 &= \vec{b}_2 > \text{may depend on previous solns} \\ A \vec{x}_3 &= \vec{b}_3 \quad \text{adaptive algos?} \end{aligned}$$

Even then, could solve systems simultaneously.

$[A | \vec{b}_1 \vec{b}_2 \dots]$ like finding A^{-1}

Discuss later - 2.4
(LU-factor method is better for even this.)

You just need to find A^{-1} once.
See p. 76

Using Software, Theory

2.3.9

(CAN SKIP)

Sensitivity Analysis $A\vec{x} = \vec{b}$, what happens to \vec{x} (solution) if you jiggle \vec{b} ?

Meyer 33-34

 $\vec{x} = ?$

$$\text{Old Ex } \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} -3 \\ 2 \end{bmatrix}$$

In an ill-conditioned system, the change is huge.
(Near-II structures)

$$\text{"Jiggle" } \vec{b} \quad \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4.1 \\ 2.9 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 4.1 \\ 2.9 \end{bmatrix}$$

$$= \begin{bmatrix} -2.2 \\ 1.7 \end{bmatrix}$$

what does A^{-1} mean? 6.20

$$\begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 4.2 \\ 2.8 \end{bmatrix}$$

$$= \begin{bmatrix} -1.4 \\ 1.4 \end{bmatrix}$$

$$\begin{bmatrix} 4+6k \\ 3-6k \end{bmatrix} \rightarrow$$

small change
change linearly

$$\vec{x} = \begin{bmatrix} 3x_1 - 5x_2 + 18k \\ -x_1 + 2x_2 - 3k \end{bmatrix}$$

User:
 Interesting
 Good for pts
 Good for manip
 many mats
 Heron's way.
 LU-factor
 approach to
 solving
 systems

2.4: ELEMENTARY MATRICES

① EA

An $m \times m$ matrix is elementary \Leftrightarrow
 It can be obtained from I_m after just one ERO.

The elementary matrix "represents" this ERO.

$$\text{Ex } I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \leftarrow \cdot 4$$



$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

elementary representing
 "E" $4 \cdot R_3 \rightarrow R_3$

If A is $m \times n$, multiplying A by E
on the left applies the ERO
 represented by E .

$$\text{If } A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$$

$$\text{Then, } EA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ \underline{20} & \underline{24} \end{bmatrix}$$

\leftarrow ERO for E
was performed!

*Don't say
permutation -
often not elem!*

$$\text{Ex } \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$$

E A EA
 $(R_1 \leftrightarrow R_2)$

Note $\left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix} \right)$

A E AE
 \swarrow
(columns were
switched!)

$A_{m \times n} E_{n \times n}^{\text{ERO}}$ corresponds to column ops.
(Don't worry...)

$$\text{Ex } A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

$\text{ERO: } R_2 + (-4)R_1 \rightarrow R_2$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \end{bmatrix} \star$$

Find an elementary matrix E such that
 $EA = (\text{this new matrix}) \star$

To construct E , take I_m
 (here I_2 , since $A \in \mathbb{R}^{2 \times 3}$)
 and perform the same ERO
 on it.

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

\downarrow
 apply same ERO

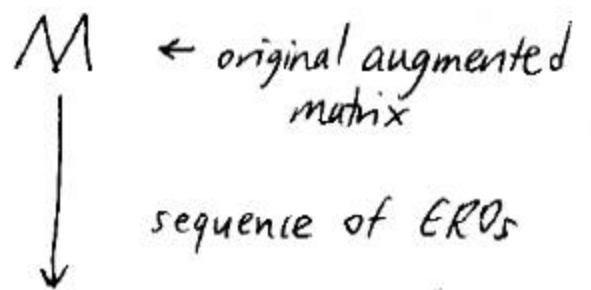
$$E = \begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix}$$

✓ Then, $EA = \begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$

$$= \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \end{bmatrix} \star$$

③ Gaussian Elimination

Book uses A -
often repr.
coeff. matrix



$N \leftarrow$ row-echelon form

Idea
early

Ex 3 (p. 81)

Write M 1st.

$$N = E_3 E_2 E_1 M$$

$\xleftarrow{\text{apply}}$
 corresp.
 EROs
 right-to-left

Up to #12

④ E^{-1}

Any elementary matrix E is invertible,
and E^{-1} is also elementary.

E^{-1} represents the ERO that undoes
the ERO for E .

$$\begin{array}{c} [E | I] \\ \downarrow \\ [I | E^{-1}] \end{array}$$

2.4.5

Ex
$$\left[\begin{array}{ccc|ccc} 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{array} \right]$$

E I

represents
 $R_1 \leftrightarrow R_3$

$\downarrow R_1 \leftrightarrow R_3$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{array} \right]$$

I

$E^{-1} = E$
for a row
interchange

Ex
$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 4 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array} \right] \xleftarrow{-\frac{1}{4}}$$

E I

represents
 $4 \cdot R_2 \rightarrow R_2$

$\left(\begin{matrix} 1 \\ 0 \\ 0 \end{matrix} \right)$ on main diag. on left
 $\Rightarrow \left(\begin{matrix} 1 \\ 0 \\ 0 \end{matrix} \right)$ on right

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array} \right]$$

E^{-1} represents
 $\frac{1}{4}R_2 \rightarrow R_2$

Ex $\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array} \right]$

E represents
 $R_2 + 3 \cdot R_1 \rightarrow R_2$

E^{-1} represents
 $R_2 + (-3) \cdot R_1 \rightarrow R_2$

$(c \neq 0)$ off
main diag. on left
 $\Rightarrow (-c)$ on right

① LU-Decompositions/Factorizations of A

Sometimes, a square matrix A can be factored as

$$A = \begin{matrix} L & U \\ \downarrow & \downarrow \\ \text{lower} & \text{upper} \\ \text{triangular} & \end{matrix}$$

$$\left[\begin{smallmatrix} m & 0 \\ a_{11} & n \end{smallmatrix} \right] \quad \left[\begin{smallmatrix} m & a_{12} & \dots \\ 0 & n & \dots \end{smallmatrix} \right]$$

(using
EROS →)

A can always be reduced to a "U"
(e.g., row echelon form).

is always in U
form for a sq. matrixM103 Notes
General stmt:
If allow:
 $PA = LU$

If you can do this w/out switching rows,
then you can find an LU-fac'n.

Finding L is easier if we don't rescale rows
(unnecessary).

forwardelim.

We will $A \rightarrow U$ using only row replacements. []

Fix columns left-to-right.

Ex $A = \begin{bmatrix} 2 & 3 & 0 \\ -2 & -2 & -2 \\ 0 & 3 & -2 \end{bmatrix}$ $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$R_2 + R_1 \rightarrow R_2$

$\begin{bmatrix} 2 & 3 & 0 \\ 0 & 1 & -2 \\ 0 & 3 & -2 \end{bmatrix}$ $E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$R_3 + (-3)R_2 \rightarrow R_3$

$\underbrace{\begin{bmatrix} 2 & 3 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 4 \end{bmatrix}}_{\text{can be } U}$ $E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix}$

Then, L is

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix}$$

P. 86 - true in general
Never 142
Works if the "forwardelim" L-to-R!

Write I , except write the opposites of the circled entries (in the E_i 's) in the corrsp. positions.

Why?

$$U = (E_2 E_1) A$$

The product of ^{same-size} invertible matrices is invertible. (HW 2.3 #38a)

$$(E_2 E_1)^{-1} U = \underbrace{(E_2 E_1)^{-1}}_{I} (E_2 E_1) A$$

inv. of product
= reverse product of invs.

$$\underbrace{E_1^{-1} E_2^{-1}}_L U = A$$

$$L = E_1^{-1} E_2^{-1}$$

$$= \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \right) \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix}^{-1} \right)$$

Remember, these are elem. matrices corrsp. to rowreps.

$$= \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Because of
all these 0's
there aren't
too many
interactions

Can ✓ $LU = A$

⑥ Solving Systems w/LU Fac's

So there is
exactly 1 soln

Solve $A\vec{x} = \vec{b}$ (assume A invertible)
 $L\vec{U}\vec{x} = \vec{b}$ (if we can)

Let $\vec{U}\vec{x} = \vec{y}$ ← Solve 2nd for \vec{x}
 $L\vec{y} = \vec{b}$ ← Solve 1st for \vec{y}

Ex Solve $\begin{bmatrix} 2 & 3 & 0 \\ -2 & -2 & -2 \\ 0 & 3 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -8 \\ 2 \\ -14 \end{bmatrix}$

$$A \quad \vec{x} = \vec{b}$$

$$A = LU$$

(we found)

① Solve $L\vec{y} = \vec{b}$

already in
nicle, triangular
form

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -8 \\ 2 \\ -14 \end{bmatrix}$$

→ System

A way to
remember
how to do
the LU Method.

$$\begin{array}{lcl} y_1 & = -8 \\ -y_1 + y_2 & = 2 \\ 3y_2 + y_3 & = -14 \end{array}$$

Forward
subs.
1st

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -8 \\ -6 \\ 4 \end{bmatrix}$$

② Solve $U\vec{x} = \vec{y}$

$$\begin{bmatrix} 2 & 3 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -8 \\ -6 \\ 4 \end{bmatrix}$$

→ System
Use back subs.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \\ 1 \end{bmatrix}$$

(Can ✓ in original system)

Why?

At core of best modern algorithms

Systems w/same A , different \vec{b} 's

(Ch. 1 methods: have to store, manipulate simultaneously - you may not even know some of the \vec{b} 's yet.)

$\vec{x} = A^{-1}\vec{b}$ method slower, less accurate

Idea: $[A|\vec{b}_1 \vec{b}_2]$

$$\xrightarrow{GJ} [I|k_1 k_2]$$

Recipes: (1) 3x slower
GE 2x
for 1 system??
sensitivity analysis

F The Invertible Matrix Theorem Key

Assume A is $n \times n$.

The following are equivalent:

① A is invertible.

② $A\vec{x} = \vec{b}$ has 1 unique sol'n for every $\vec{b}_{(n \times 1)}$

③ $A\vec{x} = \vec{0}$ has only the trivial sol'n, $\vec{x} = \vec{0}$.

④ A is row-equivalent to I_n . ($A \sim I_n$) $\begin{pmatrix} CA|I_n \\ \text{ZERO} \\ CI|A^{-1} \end{pmatrix}$

⑤ A can be written as the product of elem. matrices. (theory)

Later: $\det(A) \neq 0$

If one is true \Rightarrow all are true.

be a strong
of lights...

2.4.12

Your book
doesn't do
this concisely.

Why?

$$\textcircled{1} \Rightarrow \textcircled{2}: A\vec{x} = \vec{b}$$

$$\vec{x} = A^{-1}\vec{b}$$

is the unique sol'n

Me: cf 2.3 \textcircled{10}

p.75

$$\textcircled{2} \Rightarrow \textcircled{3}: \textcircled{2} \Rightarrow A\vec{x} = \vec{0} \text{ has 1 unique sol'n}$$

homog. $\Rightarrow \vec{x} = \vec{0}$ is a sol'n

How many
free vars?

$$\textcircled{3} \Rightarrow \textcircled{4}: \textcircled{3} \Rightarrow \text{No free vars.}$$

$$\left[A \mid 0 \right] \xrightarrow[\text{form}]{\text{RRE}} \left[\begin{matrix} 1 & 0 & | & 0 \\ 0 & 1 & | & 0 \\ 0 & 0 & | & 0 \end{matrix} \right]$$

$A \sim I_n$

every col has a leading 1.

$$\textcircled{4} \Rightarrow \textcircled{5}: \textcircled{4} \Rightarrow A \sim I_n$$

$A = I_n$ or $A = I_n^{\text{elem. form}}$

$(E_1 \dots E_r E_s) A = I_n$ apply EROs

$$A = (E_1 \dots E_r E_s)^{-1}$$

$$A = E_s^{-1} E_r^{-1} \dots E_1^{-1}$$

shows how also elem.

$(B^{-1})AB$

We've proven
2.0 There's one
 $(\{1\} \cdot 2)$

$$\textcircled{5} \Rightarrow \textcircled{1}: \textcircled{5} \Rightarrow A \text{ is the product of invertible matrices}$$

$\Rightarrow A \text{ is invertible}$

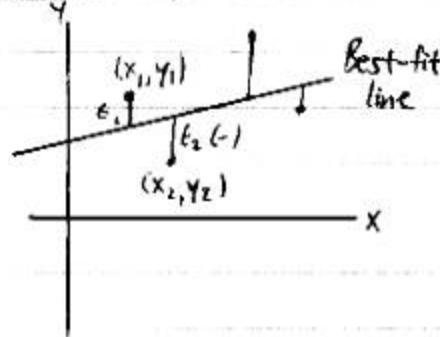


2.5: LEAST SQUARES REGRESSION ANALYSIS (p. 99) Stats!

If had 6 points,
would we try to
fit a quartic?

for modeling
the data

See Ex 10 (p. 102)



We want the linear model

$y = a_0 + a_1 x$
that minimizes the sum
of squared errors ($\sum \epsilon_i^2$).
($\sum \epsilon_i = 0$)

We need $\begin{bmatrix} a_0 \\ a_1 \end{bmatrix} \leftarrow "A"$

Let $X = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix}$
 ↑
 "1's" ↑
 plug in
 x coords.

x_i
don't need
to be sorted

be consistent

and $Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$
 ↑
 plug in
 y coords.

Then, $A = \underbrace{(X^T X)^{-1}}_{\text{1st}} \underbrace{X^T Y}_{\text{2nd (vector)}} \quad \text{vector of coeffs.}$ (If nec, I'll give you
 $\underbrace{\quad}_{\text{3rd}}$ $\underbrace{\quad}_{\text{4th}}$ this on a test)

It's got
everything:
matrix mult.,
 T^{-1}