What does A^{-1} mean? 6.3 E

2.3: A^{-1} (A Inverse)

$3 \cdot \frac{1}{3} = 1$

mult. inverses; mult. id.

A Definition

Assume $A, B, C, \ldots$ are square $n \times n$ matrices.

What are the analogs in matrix algebra?

A has either <u>no inverses</u> or <u>exactly one</u> (A^{-1}).

$\Downarrow$ A is <u>noninvertible</u> or <u>singular</u>

$\Downarrow$ A is <u>invertible</u> or <u>nonsingular</u>.

If A is invertible, there is a <u>unique</u> $n \times n$ matrix A^{-1} that satisfies:

$$AA^{-1} = I_n$$
$$\text{and } A^{-1}A = I_n$$

If one holds $\Rightarrow$ both hold.

To show $B = A^{-1}$, just show $AB = I_n$.
Then, $BA = I_n$ is automatic.

<u>Note</u>: In other books, <u>nonsquare</u> matrices may have left, right inverses.

IGNORE

NO!!

Schaum 4.21 p. 114
$AB = I$
A not inv'e $\rightarrow$ $A \neq I$
$A \sim$ mat w/ 0 row
$\underbrace{E_k \cdots E_1 AB}_{I}$
which is inv'e has a 0 row
$\times$
$\therefore A$ inv'e

Hoffman p. 24
$BA = I$
$\rightarrow AX = 0$ has only $\vec{0}$ sol'n
since $BAX = 0$, $X = 0$
$A \sim I$ etc.

<u>Proof</u> (p. 67) – uniqueness

B – right inverse of A

C – left

$$AB = I$$
$$\underbrace{CA}_{I}B = CI$$
$$B = C$$

Book: 2-sided inv., 2-sided inv. $\}$ same

right inv., left inv. $\}$ same; A^{-1} is unique

(B) Finding A^{-1}

Ex If $A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$, find A^{-1}.

See 2.1 #25

i.e., $\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} ? \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\approx \frac{1}{5}$ understood

Idea

$$\begin{bmatrix} A \end{bmatrix} \underbrace{\begin{bmatrix} \vec{x}_1 \mid \vec{x}_2 \end{bmatrix}}_{A^{-1}} = \left[\begin{array}{c|c} 1 & 0 \\ 0 & 1 \end{array}\right]$$

Good idea to bring up. Keep "Idea"

See 2.1 #53 (Block multiplication)

Simultaneously solve 2 systems:

$$A\vec{x}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$
$$A\vec{x}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Same coeff. matrix

Solving for cols of A^{-1}

Makes things very convenient for us!

Method 1 (Gauss-Jordan Elim.)

Ex $\left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 2 & 4 & 0 & 1 \end{array}\right]$ "adjoining"

Set up $[A \mid I]$

$\downarrow$ EROs

$[I \mid \text{get } A^{-1}]$

R_2 :	2	4	0	1
$+(-2)R_1$:	-2	-6	-2	0
new R_2 :	0	-2	-2	1

$$\left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & -2 & -2 & 1 \end{array}\right] \leftarrow \div(-2)$$

$$\left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & 1 & 1 & -\frac{1}{2} \end{array}\right]$$

$$R_1 + (-3)R_2 \to R_1$$

$$\left[\begin{array}{cc|cc} 1 & 0 & -2 & \frac{3}{2} \\ 0 & 1 & 1 & -\frac{1}{2} \end{array}\right] \quad A^{-1}$$

I ; $\vec{x}_1$, $A\vec{x}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$; $\vec{x}_2$, $A\vec{x}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

<u>In general</u> $(n \times n)$

$$[A \mid I_n] \xrightarrow{\text{EROs}} [I_n \mid \text{becomes } A^{-1}]$$

If you <u>ever</u> get a row of "0"s in the left square, you can't get $[I \mid A^{-1}]$, and A^{-1} doesn't exist. (A is singular).

<u>Ex</u> $\left[\begin{array}{ccc|ccc} 0 & 0 & 0 & 3 & 5 & 4 \end{array}\right]$

<u>Method 2</u> (Shortcut formula <u>Only</u> for 2×2 Matrices)

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

then the <u>determinant</u> of A

<u>not</u> abs. value

$= \det(A)$, or $|A|$

$= ad - bc$

Think: butterfly

signs

/ skew diagonal

and $A^{-1} = \dfrac{1}{\det(A)} \underbrace{\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}_{\text{adjoint of } A}$

switch

If $\det(A) = 0$, A^{-1} doesn't exist.

old

<u>Ex</u> $A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$

$\det(A) = (1)(4) - (3)(2)$
$= -2$

$A^{-1} = \dfrac{1}{-2} \begin{bmatrix} 4 & -3 \\ -2 & 1 \end{bmatrix}$

$= \begin{bmatrix} -2 & \frac{3}{2} \\ 1 & -\frac{1}{2} \end{bmatrix}$

Know both methods!

© Properties

Assume for now: inverses exist,
sizes compatible

Key

① $(A^{-1})^{-1} = A$
② $(AB)^{-1} = B^{-1}A^{-1}$

like A^T props.

Proof Show: the inverse of AB is $B^{-1}A^{-1}$.
Sufficient to show $(AB)(B^{-1}A^{-1}) = I$.

$$\begin{aligned} &(AB)(B^{-1}A^{-1}) \\ &= A\underbrace{(BB^{-1})}_{I}A^{-1} \quad \text{(Matrix mult. is assoc.)} \\ &= \underbrace{AI}_{A}A^{-1} \\ &= AA^{-1} \\ &= I \end{aligned}$$

Remember series $1-3+3-5+5-7$

"telescoping proof"

$\rightarrow\leftarrow$

cancellations

Is this B? Not nec.

Warning ABA^{-1} (Matrix mult. is not comm.)

"in the way"

③ Cancellation Props.

If C is invertible,

$AC = BC \to A = B$
$CA = CB \to A = B$ (HW)

Proof

$$\begin{aligned} AC &= BC \\ A\underbrace{CC^{-1}}_{I} &= B\underbrace{CC^{-1}}_{I} \\ A &= B \end{aligned} \qquad \left(\begin{aligned} CA &= CB \\ \underbrace{C^{-1}C}_{I}A &= \underbrace{C^{-1}C}_{I}B \\ A &= B \end{aligned} \right)$$

Also

can't invert 0

④ $(cA)^{-1} = \frac{1}{c}A^{-1}$, if $c \neq 0$

Ex $(3A)^{-1} = \frac{1}{3}A^{-1}$

✓: $(3A)(\frac{1}{3}A^{-1}) = \underbrace{(3)(\frac{1}{3})}_{1}\underbrace{(AA^{-1})}_{I} = I$

⑤ $(A^T)^{-1} = (A^{-1})^T$

The inverse and transpose ops. commute.

⑥ $\underbrace{(A^k)^{-1} = (A^{-1})^k}_{\text{call } A^{-k}}$, if k is a natural #

"Inverse" and exponentiation commute.

^{T} of a sum = sum of ^{T}s

Warning $(A+B)^T = A^T + B^T$, but
$(A+B)^{-1} \neq A^{-1} + B^{-1}$ usually

① Inverses and Systems

System $A\vec{x} = \vec{b}$ where A is square.

If A is invertible $\rightarrow$

This system has 1 unique sol'n, namely $\vec{x} = A^{-1}\vec{b}$.

Me: Is Why? suff. for uniqueness if we assume A^{-1} is unique?

If $A\vec{x}=\vec{b}$, then $\vec{x}=A^{-1}\vec{b}$. If $\vec{x} \neq A^{-1}\vec{b}$ $\rightarrow A\vec{x} \neq \vec{b}$

Uniqueness pf doesn't require uniqueness of A^{-1}.

Why? $A\vec{x} = \vec{b}$

$\rightarrow \underbrace{A^{-1}A}_{I}\vec{x} = A^{-1}\vec{b}$

$\rightarrow \vec{x} = A^{-1}\vec{b}$

(Why unique?

If $A\vec{x}_1 = \vec{b}$

$A\vec{x}_2 = \vec{b}$

$\Rightarrow A\vec{x}_1 = A\vec{x}_2$

$\Rightarrow \vec{x}_1 = \vec{x}_2$)

If A is singular $\rightarrow$

This system has either no sol'n or ∞ many.

Ex Solve $\begin{cases} 2x_1 + 5x_2 = 4 \\ x_1 + 3x_2 = 3 \end{cases}$

If A is inv'e, I could have $\vec{b} = \begin{bmatrix} 10^{24} \\ \pi \end{bmatrix}$ $\rightarrow$ 1 sol'n.

$$\underbrace{\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 4 \\ 3 \end{bmatrix}}_{\vec{b}}$$

Optional

$$\det(A) = (2)(3) - (5)(1)$$
$$= 1 \quad \leftarrow \text{not } 0, \text{ so } A \text{ is invertible}$$

Find A^{-1} (easy for 2×2)

$\frac{1}{\det(A)} = 1$

$$A^{-1} = \frac{1}{1}\begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$$

$$\vec{x} = A^{-1}\vec{b}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} -3 \\ 2 \end{bmatrix}$$

$$\begin{cases} x_1 = -3 \\ x_2 = 2 \end{cases}$$

Criticisms of $\vec{x} = A^{-1}\vec{b}$ Method

Only works when A square, invertible.

Finding $A^{-1}\vec{b}$ is less efficient for solving larger systems, (compared to Ch. 1 methods,) <u>unless</u> you want to solve several systems w/same A:

$$\left.\begin{aligned} A\vec{x}_1 &= \vec{b}_1 \\ A\vec{x}_2 &= \vec{b}_2 \\ A\vec{x}_3 &= \vec{b}_3 \end{aligned}\right\} \text{may depend on previous sol'ns}$$

adaptive algors?

Even then, could solve systems simult.

$[A \mid \vec{b}_1, \vec{b}_2 \cdots]$ like finding A^{-1}

Discuss later – 2.4

LU-fac'n method is better for even this.

You just need to find A^{-1} once.

See p. 76

Using Software, Theory

CAN SKIP

Sensitivity Analysis

$A\vec{x}=\vec{b}$ What happens to $\vec{x}$ (solution) if you jiggle $\vec{b}$?

Meyer 33-34

In an ill-conditioned system, the change is huge. (Near-ll structures)

Old Ex

$$\underset{A}{\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}} \underset{\vec{x}}{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}} = \underset{\vec{b}}{\begin{bmatrix} 4 \\ 3 \end{bmatrix}}$$

$$\underset{\vec{x}}{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}} = \underset{A^{-1}}{\begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}} \underset{\vec{b}}{\begin{bmatrix} 4 \\ 3 \end{bmatrix}}$$

$$= \begin{bmatrix} -3 \\ 2 \end{bmatrix}$$

"Jiggle" $\vec{b}$

$$\underset{A}{\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}} \underset{\vec{x}}{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}} = \underset{\text{new } \vec{b}}{\begin{bmatrix} 4.1 \\ 2.9 \end{bmatrix}}$$

$$\underset{\vec{x}}{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}} = \underset{\text{same } A^{-1}}{\begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}} \underset{\text{new } \vec{b}}{\begin{bmatrix} 4.1 \\ 2.9 \end{bmatrix}}$$

$$= \begin{bmatrix} -2.2 \\ 1.7 \end{bmatrix}$$

what does A^{-1} mean? 6.3

Me

$$\begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 4.2 \\ 2.8 \end{bmatrix} = \begin{bmatrix} -1.4 \\ 1.4 \end{bmatrix}$$

$\begin{bmatrix} 4+k \\ 3-k \end{bmatrix} \rightarrow$ soln coords change linearly

$$\vec{x} = \begin{bmatrix} 3x_1 - 5x_2 + 8k \\ -x_1 + 2x_2 - 3k \end{bmatrix}$$