Uses:
Interesting
Good for pfs
Good for manip. many mats the same way.
LU-factn approach to solving systems

2.4: ELEMENTARY MATRICES

Ⓐ EA

An $m \times m$ matrix is elementary $\leftrightarrow$
It can be obtained from I_m after just one ERO.

The elementary matrix "represents" this ERO.

Ex $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \leftarrow \cdot 4$

$\downarrow$

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

elementary "E" representing $4 \cdot R_3 \to R_3$

If A is $m \times n$, multiplying A by E on the left applies the ERO represented by E.

key

If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$

Then, $EA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$

$1+0+0 \quad 2+0+0$

$= \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 20 & 24 \end{bmatrix}$ ← ERO for E was performed!

If you have a ton of mats w/3 rows and you have to...

Ex $\underset{E}{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}} \underset{A}{\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}} = \underset{EA}{\begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}}$

$(R_1 \leftrightarrow R_2)$

Don't say permutation - often not elem!

Note $\left(\underset{A}{\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}} \underset{E}{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}} = \underset{AE}{\begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}} \right)$

(columns were switched!)

$\underset{m \times n}{A} \; \underset{n \times n}{E}^{ECO}$ corresponds to <u>column</u> ops.

(Don't worry...)

Ex $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$

ERO: $R_2 + (-4)R_1 \rightarrow R_2$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \end{bmatrix} ☆$$

Find an elementary matrix E such that
$EA =$ (this new matrix)☆

To construct E, take I_m
(here I_2, since $\underset{2\times 2}{E}\ \underset{2\times 3}{A}$)
and perform the same ERO
on it.

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

apply same ERO

$$E = \begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix}$$

✓ Then, $EA = \begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$

$$= \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \end{bmatrix} ☆$$

(B) Gaussian Elimination

Book uses A - often repr. coeff. matrix

$$M \leftarrow \text{original augmented matrix}$$

$$\Big\downarrow \text{ sequence of EROs}$$

$$N \leftarrow \text{row-echelon form}$$

Idea easy

Ex 3 (p. 81)

Write M 1st.

$$N = \underbrace{E_3 E_2 E_1}_{\substack{\longleftarrow \\ \text{apply corresp. EROs} \\ \text{right-to-left}}} M$$

Up to #12

(C) E^{-1}

Any elementary matrix E is invertible, and E^{-1} is also elementary.

E^{-1} represents the ERO that undoes the ERO for E.

$$[E \mid I]$$
$$\downarrow$$
$$[I \mid E^{-1}]$$

Ex

$$\left[\underbrace{\begin{matrix}0&0&1\\0&1&0\\1&0&0\end{matrix}}_{E}\;\middle|\;\underbrace{\begin{matrix}1&0&0\\0&1&0\\0&0&1\end{matrix}}_{I}\right]$$

E represents $R_1 \leftrightarrow R_3$

$$\Big\downarrow R_1 \leftrightarrow R_3$$

$$\left[\underbrace{\begin{matrix}1&0&0\\0&1&0\\0&0&1\end{matrix}}_{I}\;\middle|\;\underbrace{\begin{matrix}0&0&1\\0&1&0\\1&0&0\end{matrix}}_{E^{-1}=E}\right]$$

$E^{-1} = E$ for a row interchange

Ex

$$\left[\underbrace{\begin{matrix}1&0&0\\0&4&0\\0&0&1\end{matrix}}_{E}\;\middle|\;\underbrace{\begin{matrix}1&0&0\\0&1&0\\0&0&1\end{matrix}}_{I}\right] \leftarrow \cdot\tfrac{1}{4}$$

E represents $4 \cdot R_2 \to R_2$

$c \neq 1$ on main diag. on left $\Rightarrow \frac{1}{c}$ on right

$$\downarrow$$

$$\left[\begin{matrix}1&0&0\\0&1&0\\0&0&1\end{matrix}\;\middle|\;\underbrace{\begin{matrix}1&0&0\\0&\frac{1}{4}&0\\0&0&1\end{matrix}}_{E^{-1}}\right]$$

E^{-1} represents $\frac{1}{4}R_2 \to R_2$

Ex $$\left[\begin{array}{ccc|ccc}1&0&0&1&0&0\\ 3&1&0&0&1&0\\ 0&0&1&0&0&1\end{array}\right]$$

E represents $R_2+3\cdot R_1\to R_2$

$c\neq 0$ off main diag. on left $\Rightarrow$ $-c$ on right

$$\left[\begin{array}{ccc|ccc}1&0&0&1&0&0\\ 0&1&0&-3&1&0\\ 0&0&1&0&0&1\end{array}\right]$$

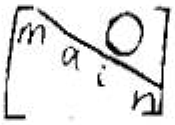
E^{-1} represents $R_2+(-3)\cdot R_1\to R_2$

① LU-Decompositions/Factorizations of A

Sometimes, a square matrix A can be factored as

$$A = L\ U$$

lower triangular   upper,



A can __always__ be reduced to a "U" (using EROs →)
(e.g., row echelon form).

is always an U form for a sq. matrix

M103 Notes
General stmt:
If allow, $PA = LU$

__If__ you can do this w/out switching rows, then you can find an LU-fac'n.

Finding L is easier if we don't rescale rows (unnecessary).

forward elim.

We will $A \to U$ using __only__ row replacements. [ ]

Fix columns left-to-right. →

__Ex__

$$A = \begin{bmatrix} 2 & 3 & 0 \\ -2 & -2 & -2 \\ 0 & 3 & -2 \end{bmatrix} \qquad I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_2 + R_1 \to R_2$

$$\begin{bmatrix} 2 & 3 & 0 \\ 0 & 1 & -2 \\ 0 & 3 & -2 \end{bmatrix} \qquad E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_3 + (-3)R_2 \to R_3$

$$\underbrace{\begin{bmatrix} 2 & 3 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 4 \end{bmatrix}}_{\text{can be } U} \qquad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix}$$

Then, L is

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix}$$

p. 86 - true in general
Meyer 142
Works if do "forward elim" L-to-R?

Write I, except write the ~~opposites~~ opposites of the circled entries (in the "E_i's") in the corresp. positions.

<u>Why?</u>

$$U = (E_2 E_1) A$$

The product of same-size invertible matrices is invertible. (HW 2.3 38a)

$$(E_2 E_1)^{-1} U = \underbrace{(E_2 E_1)^{-1} (E_2 E_1)}_{I} A$$

inv. of product = reverse product of invs.

$$\underbrace{E_1^{-1} E_2^{-1}}_{L} U = A$$

$$L = E_1^{-1} E_2^{-1}$$

Remember, these are elem. matrices corresp. to row repls.

$$= \left(\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1}\right) \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix}^{-1}\right)$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix}$$

Because of all these 0s there aren't too many interactions

$$= \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix}$$

<u>Can ✓</u> $LU = A$

(E) Solving Systems w/LU Fac'ns

So there is exactly 1 soln

Solve $A\vec{x} = \vec{b}$ (assume A invertible)

$L\underbrace{U\vec{x}} = \vec{b}$ (if we can)

Let $U\vec{x} = \text{"}\vec{y}\text{"}$ ← Solve 2nd for $\vec{x}$

$L\vec{y} = \vec{b}$ ← Solve 1st for $\vec{y}$

<u>Ex</u> Solve $\begin{bmatrix} 2 & 3 & 0 \\ -2 & -2 & -2 \\ 0 & 3 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -8 \\ 2 \\ -14 \end{bmatrix}$

$A \quad \vec{x} = \vec{b}$

$A = \underbrace{LU}_{\text{(we found)}}$

① Solve $L\vec{y} = \vec{b}$

already in nice, triangular form

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}}_{\vec{y}} = \underbrace{\begin{bmatrix} -8 \\ 2 \\ -14 \end{bmatrix}}_{\vec{b}}$$

→ System

A way to remember how to do the LU Method

$$\begin{aligned} y_1 &= -8 \\ -y_1 + y_2 &= 2 \\ 3y_2 + y_3 &= -14 \end{aligned}$$

forward sub. 1st

⋮

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -8 \\ -6 \\ 4 \end{bmatrix}$$

② Solve $U\vec{x} = \vec{y}$

$$\underbrace{\begin{bmatrix} 2 & 3 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 4 \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} -8 \\ -6 \\ 4 \end{bmatrix}}_{\vec{y}}$$

→ System

Use back sub.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \\ 1 \end{bmatrix}$$

(Can ✓ in original system)

Why?

Larson
1 web source: "more accurate"
Recipes 36
Schaum 111
$A^{-1}\vec{b}$ - slower, less acc.
can get from LU - Recipes 44

At core of best modern algorithms

Systems w/same A, different $\vec{b}$s
(Ch. 1 methods: have to store, manipulate simultaneously - you may not even know some of the $\vec{b}$s yet.)

Idea: $[A | \vec{b}_1, \vec{b}_2 \cdots]$
GJ $\downarrow$
$[I | \vec{x}_1, \vec{x}_2 \cdots]$

$\vec{x} = A^{-1}\vec{b}$ method slower, less accurate

Recipes: (-) 3x slower
GE 2x
for 1 system ??

sensitivity analysis

(F) The Invertible Matrix Theorem (Key)

30 Thms in one
$\binom{6}{2} \cdot 2$ $\rightleftarrows$

Assume A is $n \times n$.
The following are equivalent:

1. A is invertible.
2. $A\vec{x} = \vec{b}$ has 1 unique sol'n for every $\underset{(n \times 1)}{\vec{b}}$.

$\vec{x} = A^{-1}\vec{b}$ stated on next page

3. $A\vec{x} = \vec{0}$ has only the trivial sol'n, $\vec{x} = \vec{0}$.
4. A is row-equivalent to I_n. $(A \sim I_n)$ $\begin{pmatrix} [A|I] \\ \downarrow \text{EROs} \\ [I|A^{-1}] \end{pmatrix}$
5. A can be written as the product of elem. matrices. (Theory)

Later 6. $\det(A) \neq 0$

If one is $\underset{\text{false}}{\text{true}}$ $\Rightarrow$ $\underline{\text{all}}$ are $\underset{\text{false}}{\text{true}}$.

like a string of lights...

Your book doesn't do this concisely.

Why?

$$\textcircled{1} \Rightarrow \textcircled{2}: \quad A\vec{x} = \vec{b}$$
$$\vec{x} = A^{-1}\vec{b}$$

is the unique sol'n

Me: cf 2.3(D) p.75

$$\textcircled{2} \Rightarrow \textcircled{3}: \quad \textcircled{2} \Rightarrow \underbrace{A\vec{x} = \vec{0}}_{\text{homog.}} \text{ has 1 unique sol'n}$$

homog. $\Rightarrow \vec{x} = \vec{0}$ is a sol'n

How many free vars?

$$\textcircled{3} \Rightarrow \textcircled{4}: \quad \textcircled{3} \Rightarrow \text{No free vars.}$$

$$[A \mid 0] \xrightarrow[\text{form}]{\text{RRE}} \left[\begin{matrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \end{matrix} \;\middle|\; 0\right]$$

Every col has a leading 1.

$$A \sim I_n$$

$$\textcircled{4} \Rightarrow \textcircled{5}: \quad \textcircled{4} \Rightarrow A \sim I_n$$

$A = I_n$ or apply elem! ERO's

$$(E_k \cdots E_2 E_1) A = I_n$$
$$A = (E_k \cdots E_2 E_1)^{-1}$$
$$A = \underbrace{E_1^{-1} E_2^{-1} \cdots E_k^{-1}}_{\text{also elem.}} \quad \leftarrow \text{Shows how}$$

Shows you how you can construct A by writing a Π of EM's.

$(B^{-1}A^{-1})AB$

$$\textcircled{5} \Rightarrow \textcircled{1}: \quad \textcircled{5} \Rightarrow A \text{ is the product of invertible matrices}$$
$$\Rightarrow A \text{ is invertible}$$

We've proven 20 Thms in one $\binom{5}{2} \cdot 2$ $\rightarrow$ $\leftarrow$

$$\textcircled{5} \rightarrow \textcircled{1} \rightarrow \textcircled{2} \rightarrow \textcircled{3} \rightarrow \textcircled{4} \rightarrow \textcircled{5}$$