

Meyer
 det big 1750-1900
 New minor
 Late 1600s -
 Seti Karva, Leibniz
 indep. dealt w/
 det, Cramer's,
 but not
 popularized.

CH 3: DETERMINANTS

3.1 Every square matrix A has a determinant, $\det(A)$ or $|A|$, which is a #.

(A) Shortcuts

(1x1) $A = [c]$
 $|A| = c$
not abs. value

(2x2)

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

+ -

Ex $\begin{vmatrix} 1 & 3 \\ -2 & 5 \end{vmatrix} = (1)(5) - 3(-2)$
 $= 11$

(3x3) Sarrus's Rule

① Rewrite the 1st, 2nd columns on the right.

② Add products along the 3 full diagonals $\diagdown \diagdown \diagdown$

③ Subtract $\diagup \diagup \diagup$

like 2x2 $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$

Ex

$$\begin{array}{ccc|ccc}
 \cancel{-1} & \cancel{1} & \cancel{-2} & \leftarrow & \cancel{-1} & \cancel{1} \\
 \cancel{3} & \cancel{2} & \cancel{1} & & \cancel{3} & \cancel{2} \\
 \cancel{0} & \cancel{-1} & \cancel{-1} & & \cancel{0} & \cancel{-1}
 \end{array}$$

$- (0) \quad - (1) \quad - (-3)$
 $+ (2) \quad + (0) \quad + (6)$

$$= 2 + 6 + 3 - 1$$

$$= \textcircled{10}$$

Method only for 3×3 .

Ⓑ Expanding (for general $n \times n$) by Minors / Cofactors (Know!)

Same Ex

Choose a row or column to expand along, preferably one w/ "0"s.

$$\begin{array}{|c|c|c|}
 \hline
 \textcircled{-1} & 1 & -2 \\
 \hline
 3 & 2 & 1 \\
 \hline
 0 & -1 & -1 \\
 \hline
 \end{array}$$

A

3×3 sign matrix

$$\left(\begin{array}{c} \oplus \\ \downarrow \\ \text{checkerboard} \end{array} \right)$$

$$\begin{array}{|c|c|c|}
 \hline
 \oplus & - & + \\
 \hline
 - & + & - \\
 \hline
 + & - & + \\
 \hline
 \end{array}$$

$$(-1)^{i+j}$$

As you get more practice, you'll be able to write submatrices by just looking at the original.

$$\det(A) = + \underbrace{(-1)}_{\substack{\text{from} \\ \text{sign} \\ \text{matrix}}} \underbrace{1}_{\text{1st entry}} \begin{vmatrix} 1 & 1 & -2 \\ 3 & 2 & 1 \\ 0 & -1 & -1 \end{vmatrix}$$

delete row, col w/ 1

$$- (3) \begin{vmatrix} -1 & 1 & -2 \\ 2 & 1 & 1 \\ 0 & -1 & -1 \end{vmatrix}$$

$$+ (0) \begin{vmatrix} \text{who} \\ \text{cares?} \end{vmatrix} \rightarrow 0$$

$$= -1 \begin{vmatrix} 2 & 1 \\ -1 & -1 \end{vmatrix}$$

the minor M_{11}
($a_{11} = 1$)

$$-3 \begin{vmatrix} 1 & -2 \\ -1 & -1 \end{vmatrix}$$

M_{21}

$$= -1(-1) - 3(-3)$$

$$= 10$$

Coincidence

A cofactor is a minor where you incorporate the sign from the sign matrix.
 ("-" \rightarrow flip sign)

$$C_{ij} = (-1)^{i+j} M_{ij}$$

Go back:

$$\begin{cases} M_{11} = C_{11} \\ M_{21} = -C_{21} \\ C_{11} = M_{11} \\ C_{21} = -M_{21} \end{cases}$$

© Ex

$$\begin{vmatrix} 1 & -2 & 5 & 2 \\ 0 & 0 & 3 & 0 \\ 2 & -6 & -7 & 5 \\ 5 & 0 & 4 & 4 \end{vmatrix} \leftarrow \text{Expand along}$$

A

$$\begin{vmatrix} + & - & + \\ & & \ominus \end{vmatrix} \quad \text{or} \quad (-1)^{i+j}$$

$$(-1)^{2+3} = -1$$

row 2
col 3
for 3

$$|A| = - (3) \begin{vmatrix} 1 & -2 & 2 \\ 2 & -6 & 5 \\ 5 & 0 & 4 \end{vmatrix} \quad \text{Recursive}$$

Sarrus or Expand
 $\rightarrow 2$

You

$$= -3(2)$$

$$= \ominus 6$$

Me
 $\begin{matrix} \triangle, \triangle \\ \downarrow \quad \downarrow \\ \triangle \quad \triangle \end{matrix}$
 but row
 rearrangings
 may change
 sign of det
 (3.2)

① Det. of a Triangular Matrix (∇ or \triangle)

= product "along" main diagonal
 = $a_{11} a_{22} \dots a_{nn}$

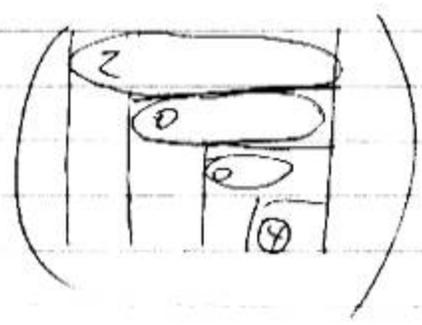
Ex $\begin{vmatrix} 1 & 3 & 0 \\ 0 & 5 & 2 \\ 0 & 0 & 4 \end{vmatrix} = (1)(5)(4) = \textcircled{20}$
 upper tri.

Idea $\begin{vmatrix} 1 & 3 & 0 \\ 0 & 5 & 2 \\ 0 & 0 & 4 \end{vmatrix} \begin{vmatrix} + & & \\ & + & \\ & & + \end{vmatrix}$
 $(1)(5)(4)$
 (Exploiting "0"s like crazy.)

+ (1) times 1
 0s don't matter

Technically
 can expand
 a 2×2 using
 cofactors

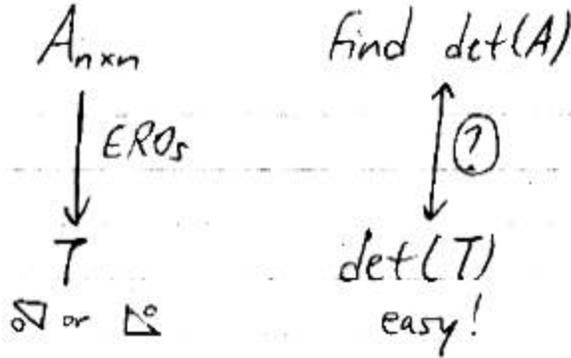
Ex $\begin{vmatrix} 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 2 & 1 & 1 & 4 \end{vmatrix} = \textcircled{0}$
 lower tri.



3.2: USING EROs, ECOs ^{Column}

(A) Idea

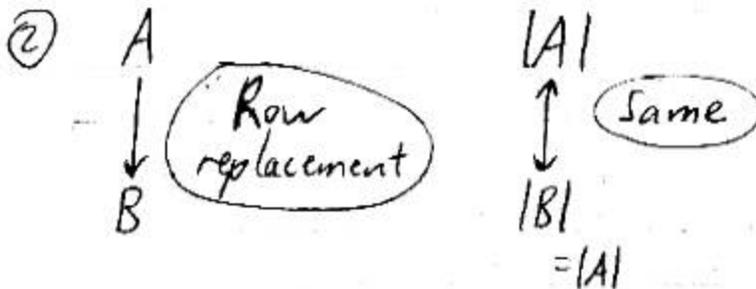
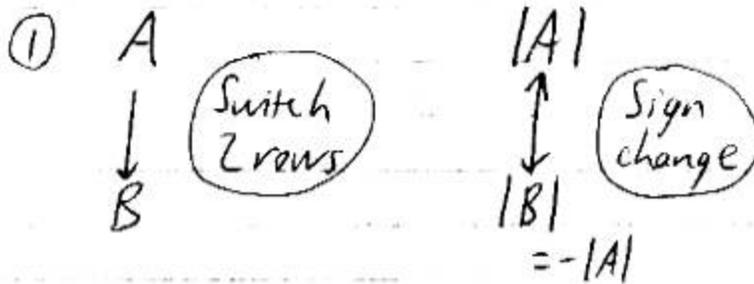
Meyer 479
Cofactor
#multy: $n! (1 + \frac{1}{2!} + \dots + \frac{1}{(n-1)!})$
 $\leq e-1$
 $e = 1 + 1 + \frac{1}{2!} + \dots$



Computer Science Note
Usually faster than 3.1 $\leftarrow O(n!)$ efficiency esp. for large n

(B) How Do EROs Affect a Det.?

Assume A, B are $n \times n$.



$$\textcircled{3} \quad \begin{array}{l} A \\ \downarrow \\ B \end{array} \quad \begin{array}{l} \text{Mult. a row} \\ \text{by } c \neq 0 \end{array} \quad \begin{array}{l} |A| \\ \downarrow \textcircled{c} \\ |B| \\ = c|A| \end{array}$$

$$\text{Ex} \quad \begin{array}{c} \left| \begin{array}{cc} 1 & 3 \\ 1 & 8 \end{array} \right| \\ \hline 5 \end{array} \xrightarrow{\cdot 2} \begin{array}{c} \left| \begin{array}{cc} 2 & 6 \\ 1 & 8 \end{array} \right| \\ \hline 10 \end{array}$$

We apply $\textcircled{3}$ as a factoring trick.

$$\text{Ex} \quad \left| \begin{array}{cc} 2 & 6 \\ 1 & 8 \end{array} \right| \leftarrow \begin{array}{l} \text{factor out "2"} \\ \text{from ~~one~~ row.} \end{array}$$

$$= 2 \left| \begin{array}{cc} 1 & 3 \\ 1 & 8 \end{array} \right|$$

$\rightarrow R_1: OK$ in GE

WARNING $R_2 + 3R_1 \rightarrow R_2$
 \uparrow
 not R_1 (you'd be rescaling R_1)

what you'd be doing then is that you'd be rescaling R_1 by 3 and then adding R_2 to it.

ECOs ^{column} follow similar rules. Switch 2 cols \rightarrow Sign flip in det etc

In 3.3, $\det(A) = \det(A^T)$
 cols of $A \leftrightarrow$ rows of A^T

Could force
you to
→ T than 60s

① Exs

Cofactor
expansion
would be
reasonable,
let of 0.

$$\text{Ex} \quad \begin{vmatrix} 2 & 3 & 1 \\ 6 & 9 & 6 \\ 0 & 4 & 3 \end{vmatrix} \leftarrow \text{factor "3"}$$

$$= 3 \begin{vmatrix} 2 & 3 & 1 \\ 2 & 3 & 2 \\ 0 & 4 & 3 \end{vmatrix}$$

$$R_2 - R_1 \rightarrow R_2$$

no change in det

$$= 3 \begin{vmatrix} 2 & 3 & 1 \\ 0 & 0 & 1 \\ 0 & 4 & 3 \end{vmatrix}$$

$$R_2 \leftrightarrow R_3$$

sign change

$$= -3 \begin{vmatrix} 2 & 3 & 1 \\ 0 & 4 & 3 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= -3(8)$$

$$= \boxed{-24}$$

Ex

$$\begin{vmatrix} 1 & 2 & -1 & 1 \\ 3 & 4 & -2 & 0 \\ 0 & 3 & 0 & -1 \\ -3 & 0 & 1 & 1 \end{vmatrix}$$

How can we turn this 3 into a 0 without disturbing the 2 '0's'?

$$C_2 + 3 \cdot C_4 \rightarrow C_2$$

$$\begin{array}{r} C_2: 2 \quad 4 \quad 3 \quad 0 \\ + 3 \cdot C_4: 3 \quad 0 \quad -3 \quad 3 \\ \hline \text{new } C_2: 5 \quad 4 \quad 0 \quad 3 \end{array}$$

no change in det.

$$= \begin{vmatrix} 1 & 5 & -1 & 1 \\ 3 & 4 & -2 & 0 \\ 0 & 0 & 0 & -1 \\ -3 & 3 & 1 & 1 \end{vmatrix} \leftarrow \text{Expand along}$$

$$= \underbrace{(-1)^{3+4}}_{(-1)^{(+)} \leftarrow \text{sign matrix}} \underbrace{(-1)}_{=4} \begin{vmatrix} 1 & 5 & -1 \\ 3 & 4 & -2 \\ -3 & 3 & 1 \end{vmatrix} \left(\begin{array}{l} \leftarrow \text{use} \\ \text{Cofactor} \\ \text{Summation} \\ \text{or} \\ \text{reduce to } \nabla \end{array} \right)$$

$$= \textcircled{4}$$

If a matrix has entries that are all ints \rightarrow is the det guaranteed to be int?

Yes - cofactor expansion
we +, -, x integers.
No ∇ , unless you factor
3.3 #43 (ii)

⑤ When can we automatically say $\det(A) = 0$?

A
↓
EROs,
ECOs
↑

At any time...

① If there is an all-0 row (or column)

$$\text{Ex } \begin{vmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 4 & 5 & 6 \end{vmatrix} \leftarrow \text{Expand along} \\ = 0$$

② If 2 rows (or 2 cols.) are equal

$$\text{Ex } \begin{vmatrix} 1 & 1 & 4 \\ 2 & 2 & 5 \\ 3 & 3 & 6 \end{vmatrix} = 0$$

$$\left(\begin{array}{c} C_1 - C_2 \rightarrow C_1 \\ \begin{vmatrix} 0 & 1 & 4 \\ 0 & 2 & 5 \\ 0 & 3 & 6 \end{vmatrix} \\ = 0 \end{array} \right)$$

includes 0, ②
 ③ If 1 row is a multiple of another, row
 (or col.)

$$\text{Ex } \begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 0 & 0 & 1 \end{vmatrix} (\leftarrow R_2 - 2 \cdot R_1) = 0$$

$$\left(\begin{array}{c} = \begin{vmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} \\ = 0 \end{array} \right)$$

Even w/out these, $\det(A)$ could = 0.

$$\begin{array}{cccc} 1 & 2 & 0 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 12 \\ 0 & 0 & 0 & 22 \\ 0 & 0 & 0 & 24 \end{array}$$

p. 110.
 Must be equiv.
 to a matrix
 w/ a 0-row.

(\downarrow makes 4
 rows)

3.3: PROPERTIES OF DETS.

A, B are $n \times n$.

Proof in book
involves elem.
matrices,
induction

$$\textcircled{1} |AB| = |A||B|$$

det of product = product of det's

$$\textcircled{2} |cA| = c^n |A|$$

scalar
why?

$$A = \begin{bmatrix} | & a_{11} & | \\ | & a_{12} & | \\ \vdots & \vdots & \vdots \\ | & a_{n1} & | \end{bmatrix}$$

$$|cA| = \begin{bmatrix} | & ca_{11} & | \\ | & ca_{12} & | \\ \vdots & \vdots & \vdots \\ | & ca_{n1} & | \end{bmatrix} \quad \left. \begin{array}{l} \updownarrow \\ \updownarrow \end{array} \right\} \begin{array}{l} \text{factor "c"} \\ n \text{ times} \end{array}$$

$$= c^n \begin{bmatrix} | & a_{11} & | \\ | & a_{12} & | \\ \vdots & \vdots & \vdots \\ | & a_{n1} & | \end{bmatrix}$$

$$= c^n |A|$$

Up to #9

③ A is invertible $\Leftrightarrow |A| \neq 0$

In 2.4 (F), equivalent:

A is invertible
 $A\vec{x} = \vec{b}$; unique sol'n

$A\vec{x} = \vec{0}$; $\vec{0}$

$A \sim I_n$

$A = \text{product of EMs}$

$\det(A) \neq 0$

PF $\Rightarrow A$ is inv'e

$$\Rightarrow AA^{-1} = I$$

$$\Rightarrow |AA^{-1}| = |I|$$

$$\Rightarrow |A||A^{-1}| = 1$$

$\neq 0$

$$\text{Note: } |A^{-1}| = \frac{1}{|A|} \leftarrow \textcircled{4}$$

PF \Leftarrow (I won't ask for this proof, but pieces fair)

Assume $|A| \neq 0$

A (square)

\downarrow ERDs

B (RRE form)

$$B = I \text{ or } B = [0 \dots 0]$$

$|B| \neq 0$

$|A| \neq 0$

Contradicts 1.4.10

\downarrow
 $A \sim I$
 A is invertible.

ERDs can't turn
 a non-0 det.
 into a 0 det.

You enter the
 ring of equar.
 stmts.

④ If A is invertible $\Rightarrow |A^{-1}| = \frac{1}{|A|}$

Pf In ③

⑤ $|A| = |A^T|$

\rightarrow Can use ECOs on dets.

Pf HW #44

Up to 42

WARNING Usually, $|A+B| \neq |A| + |B|$

PROOF BY INDUCTION

Recall: An "order n " matrix is $n \times n$.

Goal: To prove a conjecture (unproven claim) about all square matrices.

We will induct on the order of a matrix.

Let $P(k)$ denote "the conjecture [is true] for all matrices of order k "

Let A be ^{all-integer} any $n \times n$ matrix ($n > 1$, fixed int.)

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \leftarrow \text{Expand}$$

$$|A| = a_{11} C_{11} + a_{12} C_{12} + \dots + a_{1n} C_{1n}$$

" a_{ij} "s are integers

Each cofactor $C_{ij} = \pm \det(\underbrace{\text{an all-integer matrix of order } n-1}_{\text{an integer by IH}})$
sign matrix

So, all " C_{ij} "s are integers.

The integers are closed under $+$, \times .

So, $|A|$ is an integer.

$P(n)$ is true.

Q.E.D. (Done!)

Think about it...
what op. do we
need to have
to perform?

Idea: if you want
to show for 7×7 ,
consider if it be
true if property
true for 6×6 ?

A was any
arbitrary
 $n \times n$ matrix

Quod erat
demonstrandum
"that which
was to be
proved".

Take
Math 245
if you
love this.

3.4: APPLICS.(A) Eigenvalues (Ch. 7)

skip
if so,
don't assign
#13, 23

(B) Finding A^{-1}

A - $n \times n$ ($n \geq 2$)

Matrix of cofactors

$$C = \begin{bmatrix} C_{11} & C_{12} & \dots & C_{1n} \\ \vdots & \vdots & & \vdots \\ C_{n1} & C_{n2} & \dots & C_{nn} \end{bmatrix}$$

Adjoint of A

$$\text{adj}(A) = C^T$$

$$A^{-1} = \frac{1}{\det(A)} \text{adj}(A)$$

Useful "in theory" and

2×2 Case

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$C = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\text{adj}(A) = C^T$$

$$= \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$A^{-1} = \frac{1}{\det(A)} \text{adj}(A)$$

$$= \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

← signs
← switch

Used to prove...

(B) Cramer's Rule for Solving Systems

$$\text{Solve } A\vec{x} = \vec{b}$$

(n×n)

$$\left[\begin{array}{c|c} A & \vec{b} \\ \hline \text{coeff.} & \text{RHS} \end{array} \right]$$

$$\text{Let } D = |A|$$

$$D_i = \left| \begin{array}{c} A, \text{ except replace column } i \\ \text{with } \vec{b} \end{array} \right|, \quad i=1, 2, \dots, n$$

sets of
altered
matrices

Solution

$$\text{If } D \neq 0 \quad x_i = \frac{D_i}{D} \quad i=1, 2, \dots, n$$

book omits

If $D=0$

If all the " D_i "s are also 0 $\Rightarrow \infty$ many solns.
o.w., $\rightarrow \emptyset$

Ex w/ 2 Eqs.

List 18

$$\begin{cases} 2x - 9y = 5 \\ 3x - 3y = 11 \end{cases}$$

$$\left[\begin{array}{cc|c} \overset{x}{2} & \overset{y}{-9} & 5 \\ 3 & -3 & 11 \end{array} \right]$$

A
RHS

$$D = |A| = \begin{vmatrix} \overset{x}{2} & \overset{y}{-9} \\ 3 & -3 \end{vmatrix} = -6 - (-27) = \textcircled{21}$$

Instead of D_x ...

$$D_x = \begin{vmatrix} \overset{\text{RHS}}{5} & \overset{y}{-9} \\ 11 & -3 \end{vmatrix} = -15 - (-99) = \textcircled{84}$$

$$D_y = \begin{vmatrix} \overset{x}{2} & \overset{\text{RHS}}{5} \\ 3 & 11 \end{vmatrix} = 22 - 15 = \textcircled{7}$$

Solutions

$$x = \frac{D_x}{D} = \frac{84}{21} = 4 \quad y = \frac{D_y}{D} = \frac{7}{21} = \frac{1}{3}$$

 $\textcircled{\{(4, \frac{1}{3})\}}$

Can ✓

Can find x w/out
finding y !!

Ex w/3 Eqs.

$$\begin{cases} x_1 + x_3 = 0 \\ x_1 - 3x_2 = 1 \\ 4x_2 - 3x_3 = 3 \end{cases}$$

$$\left[\begin{array}{ccc|c} x_1 & x_2 & x_3 & \\ \hline 1 & 0 & 1 & 0 \\ 1 & -3 & 0 & 1 \\ 0 & 4 & -3 & 3 \end{array} \right]$$

A RHS

$$D = |A| = + (1) \begin{vmatrix} -3 & 0 \\ 4 & -3 \end{vmatrix} - 0 + (1) \begin{vmatrix} 1 & -3 \\ 0 & 4 \end{vmatrix} = 9 + 4 = \textcircled{13}$$

Cofactor
exp.
easy

$$D_1 = \begin{vmatrix} \text{RHS} & x_2 & x_3 \\ \hline 0 & 0 & 1 \\ 1 & -3 & 0 \\ 3 & 4 & -3 \end{vmatrix} = + (1) \begin{vmatrix} 1 & -3 \\ 3 & 4 \end{vmatrix} = \textcircled{13}$$

$$D_2 = \begin{vmatrix} x_1 & \text{RHS} & x_3 \\ \hline 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 3 & -3 \end{vmatrix} = + (1) \begin{vmatrix} 1 & 0 \\ 3 & -3 \end{vmatrix} - 0 + (1) \begin{vmatrix} 1 & 1 \\ 0 & 3 \end{vmatrix} = -3 + 3 = \textcircled{0}$$

$$D_3 = \begin{vmatrix} x_1 & x_2 & \text{RHS} \\ \hline 1 & 0 & 0 \\ 1 & -3 & 1 \\ 0 & 4 & 3 \end{vmatrix} = + (1) \begin{vmatrix} -3 & 1 \\ 4 & 3 \end{vmatrix} = \textcircled{-13}$$

Sol'n

$$x_1 = \frac{D_1}{D} = \frac{13}{13} = 1$$

$$x_2 = \frac{D_2}{D} = \frac{0}{13} = 0$$

$$x_3 = \frac{D_3}{D} = \frac{-13}{13} = -1$$

$$\textcircled{\{(1, 0, -1)\}} \text{ Can } \checkmark$$

Solving Systems

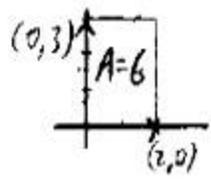
Graphing
Sub
+ Method

Gaussian, G-J Elim. } more
LU fact'n } common
A⁻¹b
Cramer's

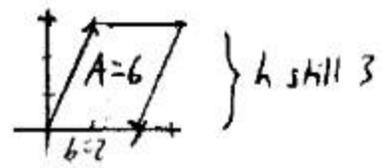
Geometry (Cool!)

Don't worry
x, y
Cramer's
|A| = |A^T|, anyway

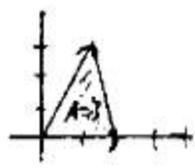
Ex $\begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix} = 6$



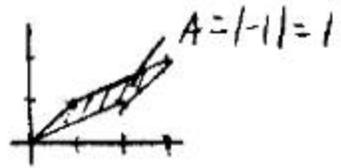
Ex $\begin{vmatrix} 2 & 1 \\ 0 & 3 \end{vmatrix} = 6$



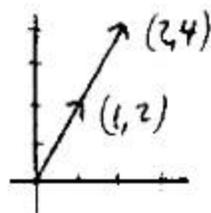
$\frac{1}{2} \begin{vmatrix} 2 & 1 \\ 0 & 3 \end{vmatrix} = 3$



Ex $\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} = -1$

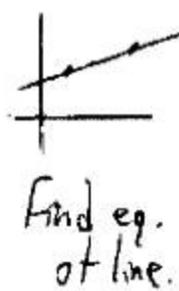
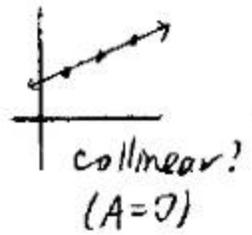
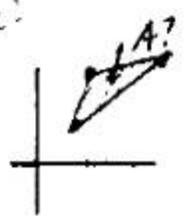


Ex $\begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} = 0$



extends to 3D parallelepiped

Book:



When do
you think
we get
A=0?