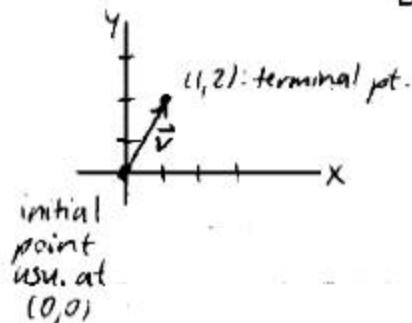


CH. 4: VECTOR SPACES4.1: VECTORS IN  $\mathbb{R}^n$ (1) Vectors in the Plane ( $\mathbb{R}^2$ )(1) DrawingBook:  $(1, 2)$ 

$$\text{Ex } \vec{v} = (1, 2) \text{ or } \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$



In principle,  
you can move  
it around.

(2) Vector "+"

$$\text{Ex If } \vec{v} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \text{ and } \vec{w} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

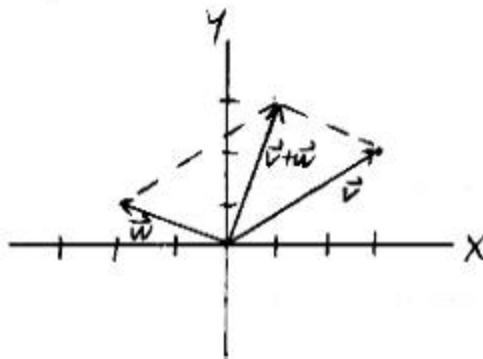
add component  
just like for  
general  
matrices

They weren't  
sure!

$$\text{then } \vec{v} + \vec{w} = \begin{bmatrix} 3 + (-2) \\ 2 + 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

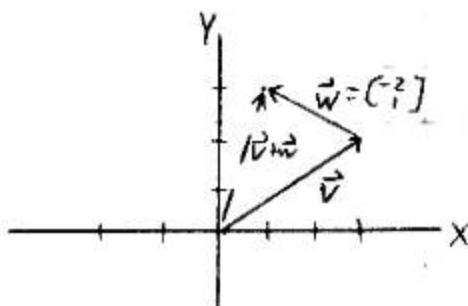
Geometrically

Parallelogram Law



$\vec{v} + \vec{w}$   
represents  
the diagonal  
of the para.  
determined by  
 $\vec{v}, \vec{w}$ , provided  
initial pt. at  $(0,0)$   
resultant

Triangle Law



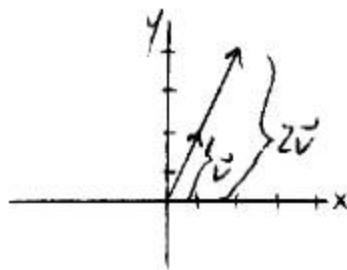
We start  
the vector  $\vec{v}$   
here...  
Head-to-tail?  
Tail-to-head?

③ Scalar Mult.

Ex If  $\vec{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

then  $2\vec{v} = \begin{bmatrix} 2(1) \\ 2(2) \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$

What do you  
think happens  
geometrically  
when you  
concent by 2?  
(Not 11, specifically)  
 $\vec{v}$  not  $\vec{v}$



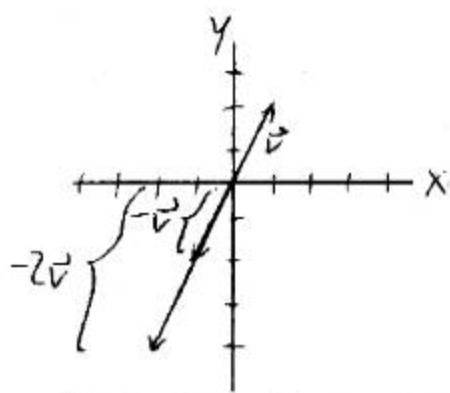
If  $c > 0$ ,  
 $c\vec{v}$  has the same direction  
as  $\vec{v}$  but is  $c$  times  
as long.

Ex If  $\vec{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

the negative of  $\vec{v}$

$$\text{then } -\vec{v} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$$

$$\text{and } -2\vec{v} = \begin{bmatrix} -2 \\ -4 \end{bmatrix}$$



If  $c < 0$ ,  
 $c\vec{v}$  and  $\vec{v}$  have  
opposite directions.

surprisingly  
important

(4)  $\vec{0}$

$$\vec{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ in } \mathbb{R}^2$$

0 multiplies  $\vec{v}$   
all the components of  $\vec{v}$

$$0\vec{v} = \vec{0}$$

$$c\vec{0} = \vec{0}$$

(Key) If  $c\vec{v} = \vec{0}$ , then  $c=0$  or  $\vec{v} = \vec{0}$

### ⑧ Vectors in $\mathbb{R}^n$

$\mathbb{R}^n = n\text{-space}$

= set of all real ordered  $n$ -tuples  
 ("points" in  $\mathbb{R}^n$ )

$$= \left\{ \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \mid \begin{array}{l} \text{"}x_i\text{"}'s are real \#s \\ \text{such that} \end{array} \right\}$$

Rules for  $\mathbb{R}^2$  extend to  $\mathbb{R}^3, \mathbb{R}^4$ , etc.

Ex If  $\vec{v} = \begin{bmatrix} 1 \\ 3 \\ -4 \\ 0 \end{bmatrix}$  and  $\vec{w} = \begin{bmatrix} 0 \\ 3 \\ 5 \\ -7 \end{bmatrix}$ .

then  $2\vec{v} - \vec{w} = ?$

$$2\vec{v} - \vec{w} = 2 \begin{bmatrix} 1 \\ 3 \\ -4 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 3 \\ 5 \\ -7 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ -8 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -3 \\ 5 \\ 7 \end{bmatrix}$$

$$= \begin{pmatrix} 2 \\ 3 \\ -13 \\ 7 \end{pmatrix} \quad \leftarrow \text{in } \mathbb{R}^4$$

## 4.2: VECTOR SPACES (VSs)

### ① What are they?

a set is  
just a  
group of  
objects

props. include  
comm.,  
dist., assoc.

A set  $V$  is a vector space  $\Leftrightarrow$   
the 10 properties on p. 171 hold.  
(axioms)

The members/elements of  $V$  are called  
vectors ( $\vec{v}, \vec{w}$ , etc.)

If it's not obvious, you must define  
vector addition ( $\vec{v} + \vec{w}$ ) and  
scalar multiplication ( $c\vec{v}$ )

The set of scalars ( $c$ , etc.) is assumed  
to be  $\mathbb{R}$  (the reals). Ch. 8:  $\mathbb{C}$  (complex #s).

### ② Examples of VSs

in chaos and  
fractal theory,  
you deal w/  
fractional  
dims.

$\vec{0} + \vec{0}$  may not =  $\vec{0}$ !

$m, n$  are fixed positive integers  
Ex  $\mathbb{R}^n$  ( $\mathbb{R}^1, \mathbb{R}^2, \mathbb{R}^3$ , etc.)

Don't mix  $\mathbb{R}^2, \mathbb{R}^3$   
 $\begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$  is undefined.

Ex  $\underbrace{M_{3,2}}_{\text{"V"}}$  = set of all real  $3 \times 2$  matrices

Vectors in  $V$  are  $3 \times 2$  matrices!!

$$\vec{v} = \begin{bmatrix} 1 & -1 \\ 3 & 0 \\ 4 & \pi \end{bmatrix} \quad \vec{w} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\vec{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

additive identity for  $V$   
For every  $\vec{v}$  in  $V$ ,  
 $\vec{v} + \vec{0} = \vec{v}$

$$-\vec{v} = \begin{bmatrix} -1 & 1 \\ -3 & 0 \\ -4 & -\pi \end{bmatrix} \quad \text{additive inverse of } \vec{v}$$

$$\vec{v} + (-\vec{v}) = \vec{0}$$

Define vector "+", scalar mult.

There are other,  
more exotic,  
ways of  
defining

Use usual matrix ops,  
(i.e., add corresp. entries for "+")

(i.e., to obtain  $c\vec{v}$ , mult. each entry of  $\vec{v}$  by  $c$ )

All 10 props. on p.171 hold

(1) Closure under vector "+"

Your book  
likes  $\vec{u}$ ,  
but we'll use  
 $\vec{v}$  later.  
I don't like  $\vec{u}$ .

We used  
these before

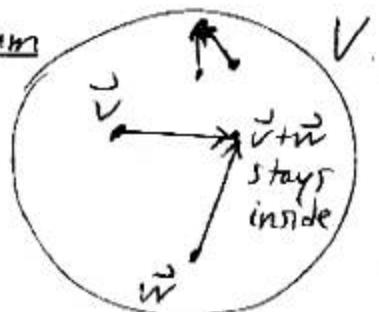
For every  $\vec{v}$  and  $\vec{w}$  in  $V$ ,  
 $\vec{v} + \vec{w}$  is in  $V$ .

Ex  $\vec{v} + \vec{w} = \begin{bmatrix} 1 & -1 \\ 3 & 0 \\ 4 & \pi \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}$

$$= \begin{bmatrix} 1 & 0 \\ 4 & 0 \\ 4 & \pi \end{bmatrix} \leftarrow \text{also in } V$$

Anytime you add two real  $3 \times 2$  matrices,  
the sum is a real  $3 \times 2$  matrix.

Venn Diagram



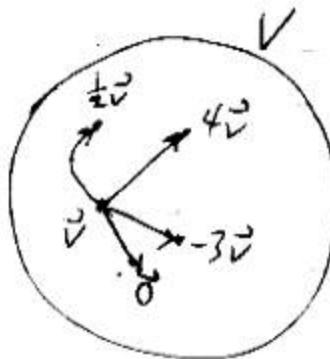
$\vec{v}$  could =  $\vec{w}$

### ⑥ Closure under scalar mult.

Then... what?

For every  $\vec{v}$  in  $V$  and every  $c$  in  $\mathbb{R}$ ,  
 $c\vec{v}$  is in  $V$ .

Ex  $\frac{1}{2}\vec{v} = \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 3 & 0 \\ 4 & \pi \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{3}{2} & 0 \\ 2 & \frac{\pi}{2} \end{bmatrix}$  also in  $V$



### ⑦, ⑧ Vector "+" is comm., assoc.

### ⑨ Additive identity

There is a  $\vec{0}$  in  $V$  such that  
 $\vec{v} + \vec{0} = \vec{v}$  for every  $\vec{v}$  in  $V$ .

$$\vec{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \text{ is in } V$$

Defn:  $\vec{v}$  = the neg. of  $\vec{v}$   
 in a VS.  
 i.e.  $\vec{v} + (-\vec{v}) = \vec{0}$   
 (V not a vec space)

"right" not  
 b/c  $(-1)\vec{v}$   
 b/c Ex 8  
 $\vec{v} = (x_1, x_2)$   
 $(-1)\vec{v} = (-x_1, 0)$

### ⑩ Additive inverse

For every  $\vec{v}$  in  $V$ , there is a vector  $-\vec{v}$  in  $V$   
 such that  $\vec{v} + (-\vec{v}) = \vec{0}$ .

### ⑪-⑫ Scalar props.

Exs of Vector SpacesEx  $M_{m,n}$  = set of all real  $m \times n$  matricesEx  $P$  = set of all polynomials in  $x$ , say

Use "standard ops." for poly. "+" and scalar mult.

$$\vec{0} = 0$$

The additive inverse of, say,

$$\vec{v} = a_2x^2 + a_1x + a_0 \text{ is}$$

$$-\vec{v} = -a_2x^2 - a_1x - a_0$$

$$\text{If } \vec{v} = 4x^3 - 2x \text{ and } \vec{w} = 2x^3 + 1$$

$$\begin{aligned} \text{then } -\vec{v} &= -4x^3 + 2x \\ 3\vec{v} &= 3(4x^3 - 2x) = 12x^3 - 6x \\ \vec{v} + \vec{w} &= 6x^3 - 2x + 1 \end{aligned} \quad )_{P}^{\text{all in}}$$

(Quickly ✓ that the 10 props. hold.)

Ex  $P_2$  = set of all polys in  $x$  of degree 2 or less  
includes 0Ex  $P_n$  = "degree  $n$  or less"

(tech has  
no degree)  
Why "or less"  
we'll see  
in a moment

Theorems for general VSSs are powerful!  
They apply to many kinds of sets!

### ③ Sets that are Not VSs

A set is a VS  $\leftrightarrow$   
all 10 props. hold for all cases

To show that  
a set is not  
a VS, you just  
need what?

One counterexample for any prop.  $\Rightarrow$   
not a VS.

assumed

I know this  
set is not  
a VS. Why not?

Ex Set of polys in  $x$  of degree 2

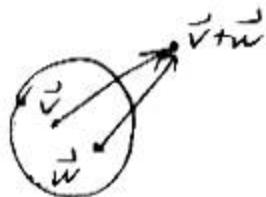
w/standard ops.

$$\text{If } \vec{v} = x^2 + x$$

$$\vec{w} = -x^2$$

$$\text{then } \vec{v} + \vec{w} = x \leftarrow \text{not in set}$$

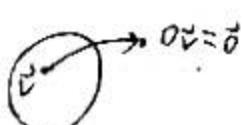
(deg=1)



So, ① Closure under "+"  $\leftarrow$  FAILS !!  
The set is not a VS.

Also, ④ fails : The set does not contain a  $\vec{0}$ .

$$\text{⑥ fails: } 0\vec{v} = \vec{0}$$



Ex The set  $\{(x,y) \mid x, y \text{ are whole } \#s\}$   
w/standard ops.

Including 0  
"whole"

Can you think  
of a counterex.  
to one of  
the axioms?  
For ex,  $(1,0)$   
does not  
have a what  
in this set?

The additive inverse of  $(1,0)$  is  $(-1,0)$ .  
but not in the set!

So, ⑤ fails. The set is not a VS.

Also, ⑦ fails:  $\frac{1}{2}(1,0) = (\frac{1}{2}, 0)$  not in the set!

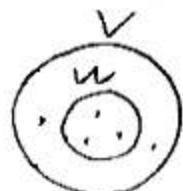
We do not have closure under scalar mult.

4.3: SUBSPACES OF VS

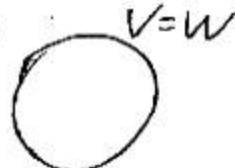
① What is a Subspace?  
Let  $V$  be a VS.

$W$  is a subset of  $V$  ( $W \subseteq V$ )

$\Leftrightarrow$  all the vectors in  $W$  are also in  $V$ .



or



$$W \subset V$$

$\uparrow$   
is a proper  
subset of  
( $W \subseteq V$ , but  $V \neq W$ )

$W$  is a subspace of  $V$   $\Leftrightarrow$

①  $W \neq \emptyset$  (empty set)

②  $W \subseteq V$

and ③  $W$  is, itself, a VS

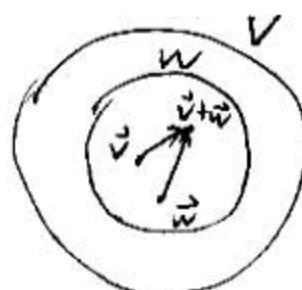
(with the same def'ns for  
vector "+" and scalar mult.  
used for  $V$  - usually assumed).

closure  
except 1, 4, 5, 6  
+ id, inv.  
Just by virtue  
of being a subset

If  $W \subseteq V$ ,  $W$  automatically inherits most of the VS props. from  $V$ . To show that  $W$  is a subspace of  $V$ , it's sufficient to show

### ① Closure<sup>of W</sup> under vector "+"

For every  $\vec{v}, \vec{w}$  in  $W \Rightarrow$   
 $\vec{v} + \vec{w}$  is also in  $W$

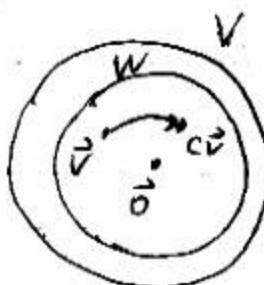


### ② Closure<sup>of W</sup> under scalar mult.

For every  $\vec{v}$  in  $W$  and every real scalar  $c \Rightarrow$   
 $c\vec{v}$  is also in  $W$

what vector

In particular,  $\vec{0}$  must be in  $W$ , ( $c=0$ )



auto inherits 6  
you want to prove  
there? the 2 remaining  
props. same w/ these

Then  $W$  will have all 10 VS props.

If  $V$  is any VS,

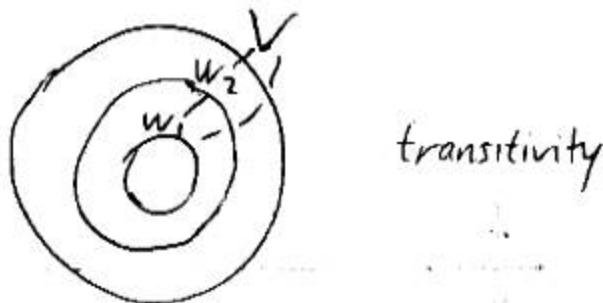
- its smallest subspace is  $\{\vec{0}\}$  (zero subspace)
- its largest subspace is  $V$
- any other subspace is a proper (non-trivial) subspace

### (B) Relating Subspaces

If  $W_1$  is a subspace of  $W_2$ , and  
if  $W_2$  is a subspace of  $V$ ,

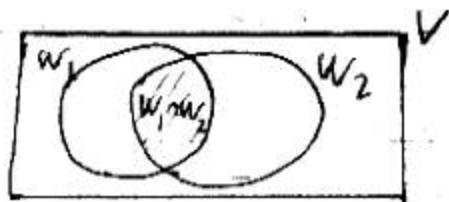
$$\Rightarrow W_1 \subset W_2 \subset V$$

$\subset$  has the same prop:  
 $a(b+c) = ab+ac$



If  $W_1$  and  $W_2$  are subspaces of  $V$   
 $\Rightarrow W_1 \cap W_2$  is also a subspace of  $V$ .

intersection



(C) Subspaces of  $\mathbb{R}^1$

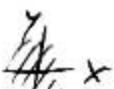
Only  $\{\vec{0}\}, \mathbb{R}^1$

Only two-  
guess  
No nontrivial  
subspaces

(D) Subspaces of  $\mathbb{R}^2$

①  $W = \{\vec{0}\}$    
(i.e.,  $\{(0,0)\}$ )

② ( $\infty$  many nontrivial subspaces)  
W = set of all "points" on a  
straight line through  $(0,0)$

③  $W = \mathbb{R}^2$  

technically,  
set of vectors  
that rep. points  
on the line

Prove ②

(lines thru  
 $(0,0)$  take  
the form  
 $y=mx$   
(y-int is 0)  
What's the  
one line  
thru  $(0,0)$   
not of this  
form?

Case 1  (slope "m" is undefined)

$$W = \{(0,y) \mid y \text{ is a real } \#\}$$

$$W \neq \emptyset, W \subseteq \mathbb{R}^2$$

Let  $\vec{v}, \vec{w}$  be arbitrary vectors in  $W$ .

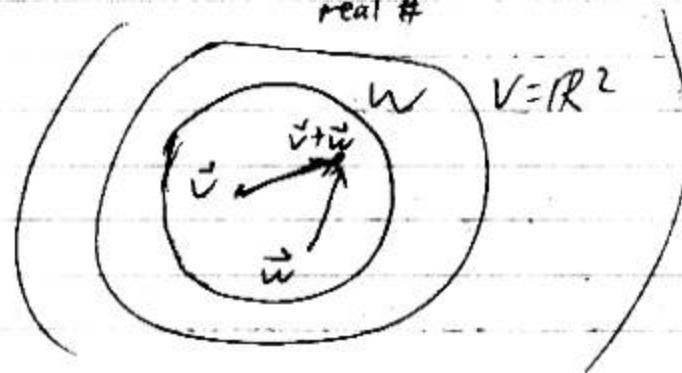
$$\begin{aligned}\vec{v} &= (0, y_1) \\ \vec{w} &= (0, y_2)\end{aligned}$$

$y_1, y_2$  are real #s

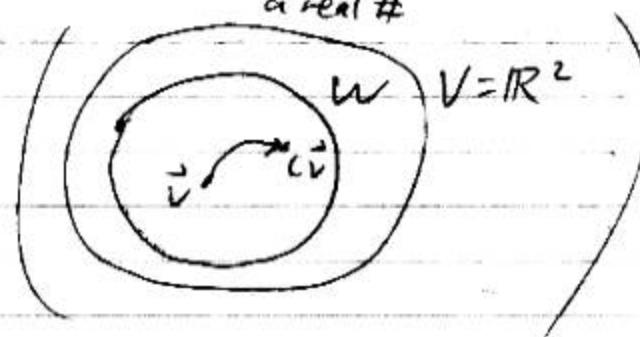
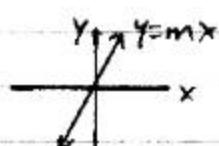
4.3.5

Prove:  $W$  is closed under vector "+"

$$\vec{v} + \vec{w} = (0, \underbrace{y_1 + y_2}_{\text{real #}}) \text{ is in } W$$

Prove:  $W$  is closed under scalar mult.If  $c$  is any real scalar,

$$c\vec{v} = (0, \underbrace{cy_1}_{\text{a real #}}) \text{ is in } W$$

Case 2On line,  $x=t$ 

$$\Rightarrow y=mt$$

$$W = \{(t, mt) \mid t \text{ is a real #}\}$$

a real #      m(same #)

$$W \neq \emptyset, W \subseteq \mathbb{R}^2$$

Let  $\vec{v}, \vec{w}$  be arbitrary vectors in  $W$ .

$$\begin{aligned}\vec{v} &= (t_1, mt_1) \\ \vec{w} &= (t_2, mt_2)\end{aligned}$$

$t_1, t_2$  are real #s

$$\vec{v} + \vec{w} = (t_1 + t_2, mt_1 + mt_2)$$

$$= (\underbrace{t_1 + t_2}_{\text{a real #}}, \underbrace{m(t_1 + t_2)}_{m(\text{same real #})})$$

or let  $t_3 = t_1 + t_2$

is in  $W$  ✓

If  $c$  is any real scalar,  $\vec{v} = (t_1, mt_1) \rightarrow$   
 $c\vec{v} = (ct_1, cmt_1)$

$$= (\underbrace{ct_1}_{\text{a real #}}, \underbrace{m(ct_1)}_{\text{same real #}})$$

a real # → same real #

is in  $W$  ✓

good proof demonstration

end of proof  
 that which was  
 to be proven  
 (Klasse war da nicht)

QED

Note: Can test  $\vec{v} + c\vec{w}$  to prove  
 closure props. simultaneously.

$$\begin{aligned}c &= 1 \rightarrow + \\ c &= 0 \rightarrow \text{m.}\end{aligned}$$

## E Subspaces of $\mathbb{R}^3$

①  $W = \{\vec{0}\}$

② <sup>(∞ many)</sup>  $W = \text{set of all "points" on a straight line through } (0,0,0)$

③ <sup>(∞ many)</sup>  $W = \text{plane through } (0,0,0)$

④  $W = \mathbb{R}^3$

dimension  
Later: "dim"  
0

1

2

3

WARNING: Is  $\mathbb{R}^2$  a subspace of  $\mathbb{R}^3$ ?

**NO**  $\mathbb{R}^2 \not\subseteq \mathbb{R}^3$

It won't work if you try to mix these - vector + is often / typically undefined

$\mathbb{R}^2$  [ ]  $\mathbb{R}^3$  [ ] Don't mix!

## F Linear Eqs.

The graph of any homog. linear eq. in n vars. represents a subspace of  $\mathbb{R}^n$ .

Ex  $2x - y + 3z = 0$

Graph: subspace of  $\mathbb{R}^3$  (Plane through  $\vec{0}$ )

generalization

The solution set of  $A\vec{x} = \vec{0}$  is a subspace of  $\mathbb{R}^n$ . (HW - #26)

Ex The sol'n set of

$$\begin{cases} x_1 + x_4 = 0 \\ x_2 - x_3 + 2x_4 = 0 \end{cases}$$

is a subspace of  $\mathbb{R}^4$ .

### ⑥ Examples

Read all Exs, except Ex 5.

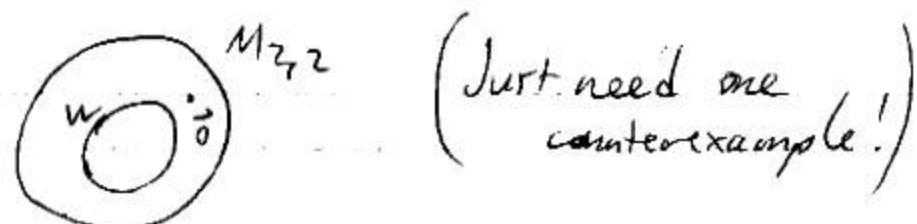
Ex  $W = \left\{ \begin{bmatrix} 1 & a \\ b & c \end{bmatrix} \mid a, b, c \text{ are real } \#s \right\}$

Is  $W$  a subspace of  $M_{2,2}$ ?

Easy ans.

(No)  $\vec{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$  is not in  $W$

The proof is

Is  $W$  closed under vector +?

or  $\begin{bmatrix} 1 & a_1 \\ b_1 & c_1 \end{bmatrix} + \begin{bmatrix} 1 & a_2 \\ b_2 & c_2 \end{bmatrix} = \begin{bmatrix} 2 & - \\ \cancel{b_1} & \cancel{c_1} \end{bmatrix}$  not in  $W$

or  $\pi \begin{bmatrix} 1 & a_1 \\ b_1 & c_1 \end{bmatrix} = \begin{bmatrix} \pi & \sim \\ \cancel{b_1} & \cancel{c_1} \end{bmatrix}$  not in  $W$

4.4: SPANNING SETS + LINEAR INDEPENDENCEReadA) Linear Combinations of VectorsLet  $V$  be a VS.Let  $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k$  be vectors in  $V$ .Then,  $\vec{w}$  is a linear combination of $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k \leftrightarrow$  $\vec{w}$  can be written astimes  $\vec{v}_1, \dots$ 

$$\vec{w} = c_1 \vec{v}_1 + c_2 \vec{v}_2 + \dots + c_k \vec{v}_k$$

where the " $c_i$ 's" are scalars.Ex Some linear combos of  $\vec{v}_1, \vec{v}_2, \vec{v}_3$  <sup>in  $\mathbb{R}^{10}$ , say</sup>:

$$3\vec{v}_1 - 4\vec{v}_2 + \vec{v}_3 \quad c_1 = 3, c_2 = -4, c_3 = 1$$

$$\pi\vec{v}_3 \quad 0 \quad 0 \quad \pi$$

$$\vec{0} \quad 0 \quad 0 \quad 0$$

what special vector

$$\text{Ex Let } \vec{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad \vec{v}_2 = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \rightarrow$$

Express  $\vec{w} = [4]$  as a linear combo of  $\vec{v}_1, \vec{v}_2$ .

Idea

Need  $c_1, c_2$  such that

$$c_1 \vec{v}_1 + c_2 \vec{v}_2 = \vec{w}$$

$$\underbrace{\begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix}}_{\text{"A"}} \underbrace{\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}}_{\substack{\text{"c"} \\ \text{weights for col. of A}}} = \vec{w}$$

Trust me!

Work

$$\text{Solve } A\vec{c} = \vec{w}$$

$$\left[ \begin{array}{cc|c} \vec{v}_1 & \vec{v}_2 & \vec{w} \end{array} \right]$$

$$\left[ \begin{array}{cc|c} -1 & 2 & 4 \\ 1 & 0 & 2 \end{array} \right]$$

$$\left[ \begin{array}{cc|c} & & \downarrow \text{Gauss-Jordan} \\ \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} & \begin{matrix} | \\ | \end{matrix} & \begin{matrix} 2 \\ 3 \end{matrix} \end{array} \right]$$

$$\begin{aligned} c_1 &= 2 \\ c_2 &= 3 \end{aligned}$$

stack the  
v vectors  
as columns  
not block mult.  
you can informally  
look at it like  
this  
 $v_1 \xrightarrow{v_2 + c_1} v_1 + c_1 v_2$

$$\vec{w} = c_1 \vec{v}_1 + c_2 \vec{v}_2$$

$$\boxed{\vec{w} = 2\vec{v}_1 + 3\vec{v}_2} \quad (\text{Ans: } [4] = [2] + [3])$$

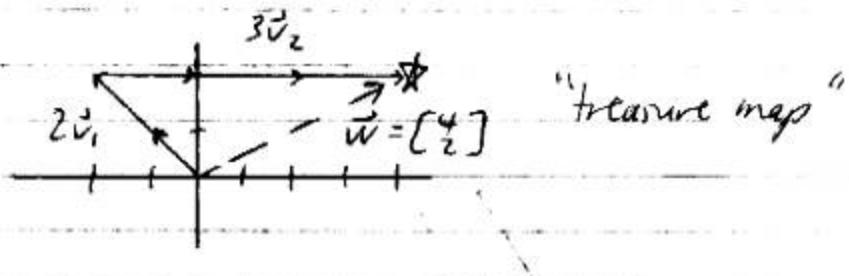
Picture

Al Bundy's  
dodge  
Treasure: 3 tablets

$$\vec{w} = 2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} + 3 \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

(111 unrelated)

2 paces  
this way,  
where it  
is a pace  
only way  
it no backtracking  
no permuting  
 $\uparrow \uparrow$



If  $A\vec{c} = \vec{w}$  is inconsistent, then  
 $\vec{w}$  can't be expressed as a linear combo  
of the " $v_i$ 's". Ex  $\cancel{\frac{v_1}{v_2}} \rightarrow \vec{w}$

If  $A\vec{c} = \vec{w}$  has  $\infty$  many sol'n's, then  
 $\infty$  many linear combos work Ex  $\cancel{\frac{v_1}{v_2}} \rightarrow \vec{w}$

Can do 1, 3a

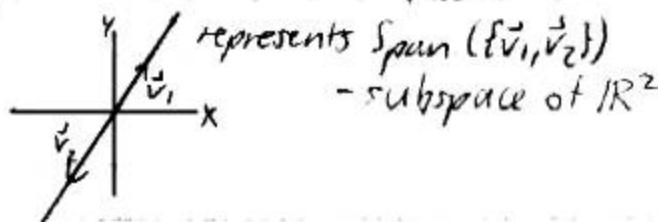
### B) Span(S)

Let  $S$  (be the set of vectors)  $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$  in  $V$

A robot  
programmed  
to walk in  
these directions.  
 $c_i$ 's can be  
any req. of real #s

Then,  $\text{Span}(S) = \text{the set of all linear combos}$   
of " $v_i$ 's" ("what we can reach")  
 $= \{c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_n\vec{v}_n \mid "c_i's \text{ are real } \#s\}$   
is a subspace of  $V$

Ex



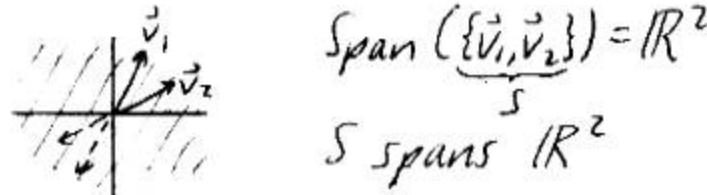
If  $\text{Span}(S) = V$ , then

"S spans V"  
"S is a spanning set of V"

Any vector in V can be written  
as a linear combo of " $v_i$ 's".

Ex

In a sense,  
we can hit  
every pt in  
the plane,  
using these  
direction vectors



" $c_i$ 's can be < 0.  
( $c_1\vec{v}_1$ )

"HW" Ex

$$S = \left\{ \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 6 \\ 2 \\ 8 \end{bmatrix} \right\}$$

Does  $S$  span  $\mathbb{R}^3$ ?  
 If not, what does  $S$  span?

$$\text{Let } A = \begin{bmatrix} 2 & 2 & 6 \\ 0 & 1 & 2 \\ 2 & 3 & 8 \end{bmatrix}$$

EROS  $\begin{bmatrix} 2 & 2 & 6 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$  row echelon  
 shape (leading 0s unnecessary)

2 pivot <sup>(PPs)</sup> positions, not 3  
 (not in book)

$S$  does not span  $\mathbb{R}^3$ .

$S$  spans a 2-dim. (<sup>subspace</sup> plane) in  $\mathbb{R}^3$ .

Why? - Later

The cols. of  $A$  span  $\mathbb{R}^m \Leftrightarrow$

each row has a <sup>(PP)</sup>

(LI: each column)

### (C) Beyond $\mathbb{R}^n$

How many entries

Under the usual ops., there is a "natural" correspondence between  $M_{m,n}$  and  $\mathbb{R}^{mn}$  and between  $P_n$  and  $\mathbb{R}^{n+1}$

Ex  $M_{2,3} = \text{set of real } 2 \times 3 \text{ matrices}$

Treat  $\vec{v} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$  in  $M_{2,3}$

as, say,  $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{bmatrix}$  in  $\mathbb{R}^6$ .

Be consistent  
if you have  
another matrix  
that's  $2 \times 3$ ,  
write the #'s  
in the same  
order.

Ex  $P_2 = \text{set of all polys in } x \text{ of degree } \leq 2 \text{ or less}$   
(includes 0)

If you +,-  
 $\mathbb{R}^3$ , or  $\mathbb{R}^m$   
 $\cdot b$ , real #s  
→ all the  
action is  
with the coeffs.

Treat  $\vec{v} = ax^2 + bx + c$  in  $P_2$

as, say,  $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$   $\begin{array}{l} \leftarrow \text{coeff. of } x^2 \\ \leftarrow \text{coeff. of } x \\ \leftarrow \text{constant term} \end{array}$  in  $\mathbb{R}^3$ .

## ① Linear Independence

The set  $S = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\}$  is a linearly independent (LI) set  $\Leftrightarrow$

$c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_k\vec{v}_k = \vec{0}$  has only the trivial sol'n  $c_1=0, c_2=0, \dots, c_k=0$   
 i.e., (if  $V=\mathbb{R}^n$ )

$$\underbrace{\begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \dots & \vec{v}_k \end{bmatrix}}_{(m \times k)} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$\vec{c}$  (weights)

has only the trivial sol'n  $\vec{c}=\vec{0}$ .

Otherwise,  $S$  is linearly dependent (LD).

Idea (in  $\mathbb{R}^n$ )

Each vector in a LI set is needed to reach new points "that couldn't be reached, otherwise. "Valuable"

If you take LGS  
of these vcs,  
the only way you  
can get 0 is  
if coeffs are  
also 0.

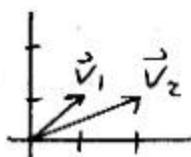
"Trust me" step

Each vector  
in the set  
is "valuable"

Ex  $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$  is a LI set.

4 Reasons:

(#1)



Each vector is "valuable" for reaching new points.

$v_1$  helps us reach points that  $v_2$  alone couldn't, and  $v_2$  helps us reach points that  $v_1$  alone couldn't.

related to Reason 1

(not II)  
If one were a scalar mult. of the other, then one would be redundant in terms of helping us reach new pts.

(#2)

A set of two vectors in  $V$  is LI  $\Leftrightarrow$  neither is a scalar multiple of the other.  
Not II.

(#3)

(General Method)

$A\vec{c} = \vec{0}$  has only the trivial soln,  $\vec{c} = \vec{0}$ .

$$\underbrace{\left[ \begin{array}{cc|c} \vec{v}_1 & \vec{v}_2 & | \vec{0} \end{array} \right]}_A$$

$$\left[ \begin{array}{cc|c} 1 & 2 & | 0 \end{array} \right]$$

$$\sim \left[ \begin{array}{cc|c} 1 & 2 & | 0 \\ 0 & -1 & | 0 \end{array} \right] \quad \begin{array}{l} \text{row-echelon shape} \\ \downarrow \text{each col. has } (pp) \\ \downarrow \text{ignore} \end{array}$$

To get to row-echelon form, just rescale row 2 by (-1), but let's look at this. I don't care if (-1) or 3, ... as long as ≠ 0

$A \neq I$ .  
steaming

that means  
system has no...

→ no free vars

→  $A\vec{c} = \vec{0}$  has only trivial sol'n

→ Columns of  $A$  ( $\vec{v}_1, \vec{v}_2$ ) form a LI set.

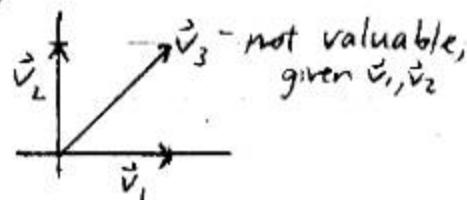
i.e.,  $c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{0}$   
 only 0s will work

(#4) ✓  $\det(A) \neq 0$ , if  $A$  is square.

Ex  $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$  is a LD set.  
 $\vec{v}_1, \vec{v}_2, \vec{v}_3$  (even though no vector is a multiple of any other!)

4 Reasons:

(#1)



(#2)

We can see:

$$\vec{v}_1 + \vec{v}_2 = \vec{v}_3$$

$$\vec{v}_1 + \vec{v}_2 - \vec{v}_3 = \vec{0}$$

"dependency relation"

one vec can  
be expressed  
as a linear  
combo of  
others

∞ many  
others  
22-2

means  $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$  has a nontrivial sol'n  
 $c_1 = 1, c_2 = 1, c_3 = -1$

## #3 (General Method)

$A\vec{c} = \vec{0}$  has nontrivial solns.

$$\underbrace{\left[ \begin{matrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & | \vec{0} \end{matrix} \right]}_A$$

already in  
RREF form!

$$\left[ \begin{matrix} 1 & 0 & | & 0 \\ 0 & 1 & | & 0 \\ \downarrow & \downarrow & \downarrow & \text{ignore} \\ 0 & 0 & | & 0 \end{matrix} \right]$$

2, not 3  $\cancel{\text{PP}}$  free var.

Related

## #4

A set of  $k$  vectors in  $\mathbb{R}^m$  where  $k > m$   
must be LD. #vecs > length or dim

$$\left[ \begin{matrix} A_{m \times n} - \text{wide} & | & \vec{0}^k \end{matrix} \right]$$

$\Rightarrow$  free vars.

(if this LLI or LD)

Ex.  $\{[0], [2] \}$  is LD

Kato Kaelin of  
vectors

$\vec{0}$  never valuable. Any set with  $\vec{0}$  is LD.

$$c_1[0] + c_2[2] = \vec{0}$$

any real # ( $c_i$ : free var.)

## 4.5: BASIS and DIMENSION

### ① What is a Basis?

Let  $V$  be a VS.

Let  $S = \underbrace{\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m\}}_{\text{in } V}$

$S$  is a basis for  $V \Leftrightarrow$

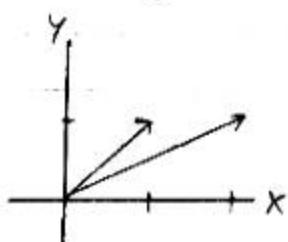
- ①  $S$  spans  $V$ , and
- ②  $S$  is LI

### Idea

A basis must have enough vectors to span  $V$ , but all of them must be "valuable." It efficiently describes  $V$ .

(If I ask for a VS, give me a basis, and I take all LCs of the basis vectors to sweep out the VS.)

Ex  $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$  is a basis for  $\mathbb{R}^2$



$$A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

Each row has a PP  
col

cols of  $A$  span  $\mathbb{R}^2$   
So, basis.

A basis for  $\mathbb{R}^m$  must have exactly  $m$  vectors.

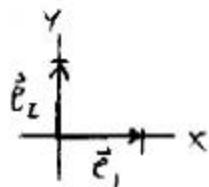
dimension of  $\mathbb{R}^m$

Sometimes if we have a vs,  
there is a very natural choice for a basis.

### (B) Standard basis for ...

$$\textcircled{R^2} \quad \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

$$\vec{e}_1 \quad \vec{e}_2$$



$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \rightarrow \begin{array}{l} \text{cols} \\ \text{span} \end{array} \rightarrow \mathbb{R}^2$$

$\downarrow$  col 1       $\downarrow$  col 2

Building up I\_m

$\textcircled{R^m}$

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \right\}$$

$$\vec{e}_1, \vec{e}_2, \dots, \vec{e}_m$$

$$A = I_m$$

$M_{m,n}$ Ex  $M_{2,3}$  (has dim = 6)

not ordered  
could do  $\binom{2+2}{2}$   
(nd)  
latter coords  
wrt basis.  
Examine ideas in D

$$\left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \right\}$$

P: How many coeffs.

(P<sub>n</sub>): polys (in x) of degree  $\leq n$ , incl. 0

$$\text{Ex } P_2: ax^2 + bx + c \quad \{1, x, x^2\}$$

$$\{1, x, x^2, \dots, x^n\}$$

up to 4

$$\dim(P_n) = n+1$$

Skip to C

(E) A vector has unique coords relative to an ordered basis.

Careful:  
coords require  
ordering.  
This is the  
order we're  
sticking with.

Let  $B = \{\vec{b}_1, \vec{b}_2, \dots, \vec{b}_m\}$  be an ordered basis for  $V$ .Each vector in  $V$  can be written in <sup>span</sup> exactly  $\checkmark$  (C) one way as a linear combo of " $\vec{b}_i$ 's.i.e., (in  $\mathbb{R}^m$ )

$A \vec{c} = \vec{w}$  has a unique sol'n  $\vec{c}$  for every  $\vec{w}$  in  $\mathbb{R}^m$ .

$\vec{b}_1, \vec{b}_2, \dots, \vec{b}_m$        $\vec{w}$  in  $\mathbb{R}^m$   
 (m x m)      coords. of  $\vec{w}$   
 rel. to B

Why we need (2) before (1)

Use Invert.  
Matrix Thm.  
Full set of  $m^2$   
linearly indep.

When is  $B$  a basis for  $\mathbb{R}^m$ ?

- $\Leftrightarrow A$  is invertible
- $\Leftrightarrow A \sim I_m \Leftrightarrow \begin{bmatrix} 1 & & & \\ 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & & & 1 \end{bmatrix}$
- $\Leftrightarrow \det(A) \neq 0$   
etc.

Ex Let  $B = \left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix} \right\}$ .

a) Show that  $B$  is a basis for  $\mathbb{R}^2$ .

① 2 vectors, not  $\parallel \rightarrow$  L.I.  
2 vectors in  $\mathbb{R}^2 \xrightarrow{\text{span}} \text{basis}$

②  $\begin{bmatrix} -1 & 2 \\ 1 & 0 \end{bmatrix} \xrightarrow{\text{square}} \begin{bmatrix} -1 & 2 \\ 0 & 2 \end{bmatrix}$

$$\begin{pmatrix} 2 & (pp) & 5 \end{pmatrix}$$

$$\left( \sim \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \right)$$

or ③  $\begin{vmatrix} -1 & 2 \\ 1 & 0 \end{vmatrix} = -2 \neq 0$

b) Write  $\vec{w} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$  as a linear combo  
of  $\vec{b}_1, \vec{b}_2$ . (4.4 ④)

Solve  $A\vec{c} = \vec{w}$

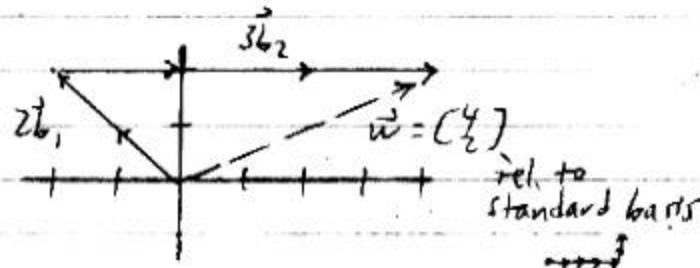
$$\left[ \begin{array}{cc|c} -1 & 2 & 4 \\ 1 & 0 & 2 \end{array} \right]$$

$\downarrow$  Gauss-Jordan

$$\left[ \begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 3 \end{array} \right]$$

coords of  $\vec{w}$   
rel. to new basis

$$\vec{w} = 2\vec{b}_1 + 3\vec{b}_2 \quad (\text{"unique basis rep."})$$



In principle,  
you can talk  
about backtracking  
or zig-zagging,  
but after you  
combine like  
terms,  
you get  
this.

This is essentially the only way  
to get  $\vec{w}$ . (after combining like terms)  
"smooths out" any  
backtracking or zig-zagging

### ① dim(V)

= dimension of  $V^{\text{vs}}$

= # of vectors in any basis for  $V$

If  $V = \{\vec{0}\}$ , then  $\dim(V) = 0$ .

Exs  $\dim(\mathbb{R}^n) = n$

$\dim(M_{m,n}) = mn$

$\dim(P_n) = n+1$

① A set of  $k$  vectors ( $k > \dim(V)$ ; "too many")  
can't be LI.

#vecs > dim(V)

Idea Not all of them can be valuable.

(We  
all VSS  
have a  
concr. tr. R  
(or C) thru  
coeffs of  
linear combns  
rel. to a basis.  
though  
opp. may  
vary.)

Ex  $\{[6], [9], [1]\}$  is clearly L0  $\nrightarrow$

$3 > 2$  or  $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$  can't have a  
 $\dim(R^2)$  "fat"  $\begin{pmatrix} p \\ p \end{pmatrix}$   
 in each col.

# trees (days of)

Why can't  
they form  
a basis?

② A set of  $k$  vectors ( $k < \dim(V)$ ; "too few") can't span  $V$ .

Ex  $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$  clearly doesn't span  $\mathbb{R}^3$

$2 < 3$  or  $\dim(\mathbb{R}^3)$  or  $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$  can't have a  $(\text{PP})$ .  
 in each row  
 "slimy"

③ A set of  $k$  vectors ( $k = \dim(V)$ ) could be a basis for  $V$ .

If you have a square matrix,  
each row has  
as many col

Need: ① They span  $V$  ↗ If you have one,  
 ② They're LI ↗ you have both if  $k$  vectors,  
 $k = \dim(V)$ .

Ex (in  $\mathbb{R}^n$ )  $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\}$  is a basis  $\iff$

$$\left[ \begin{matrix} v_1 & v_2 & \dots & v_n \end{matrix} \right] \xrightarrow{\text{Eros}} \left[ \begin{matrix} \textcircled{PP} & & & \\ & \textcircled{PP} & & \\ & & \textcircled{O} & \\ & & & \textcircled{IP} \end{matrix} \right] \sim I$$

square                          Each col, row

Up to 29

① "HW" Ex

Find a basis for  $V_{2,2}$  = VS of all upper triangular  $2 \times 2$  matrices.

Generic form:  $\left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \mid a, b, c : \text{any real } \#s \right\}$

What's a basis?

$$\begin{aligned} &= \left\{ \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & c \end{bmatrix} \mid \cdot \cdot \cdot \right\} \\ &= \underbrace{\left\{ a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \mid \cdot \cdot \cdot \right\}}_{3 \text{ basis vectors}} \end{aligned}$$

When I take  
all poss LLS

$\begin{array}{l} \text{LI} \checkmark \quad (\text{only } a=b=c=0 \Rightarrow 0_{2 \times 2}) \\ \text{Span } V_{2,2} \checkmark \end{array}$

→ go back to E

4.6: RANK and SYSTEMS $A - m \times n$ (A) Row(A)

Ex  $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 2 \\ 0 & 1 & 4 \\ 0 & 2 & 8 \end{bmatrix}$  ↪ 4 row vectors (in  $\mathbb{R}^3$ )

 $\text{Row}(A) = \underline{\text{row space of } A}$  $= \text{Span}(\text{row vectors})$   
all linear combosis a subspace of  $\mathbb{R}^n$  (here,  $\mathbb{R}^3$ )To find a basis for Row(A) $A \xrightarrow{\text{EROs}} B$  (row-echelon shape)

Take the nonzero row vectors from B. "Pivot rows"

In Ex

$$A \sim B = \left[ \begin{array}{ccc} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right]$$

 $\text{Basis for Row}(A) = \{(1, 2, 3), (0, 1, 4)\}$  $\dim(\text{Row}(A)) = 2$

Why?

If you switch rows...

If  $A \xrightarrow{\text{EROS}} B$ , then  $\text{Row}(A) = \text{Row}(B)$   
EROS do not disturb the row space.

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \leftarrow \vec{r}_1, \quad \leftarrow \vec{r}_2$$

The rank anywhere:  
neither is a  
multiple of  
the other

Why is  $\{\vec{r}_1, \vec{r}_2\}$  guaranteed to be LI?  
2 vectors (not II)

or

$$\begin{aligned} & C_1 [1 & 2 & 3] \\ & + C_2 [0 & 1 & 4] \\ & = [0 & 0 & 0] \\ & \downarrow \quad \downarrow \quad \swarrow \\ & \text{so, } c_1=0 \quad \text{so, } c_2=0 \end{aligned}$$

By definition,  $\text{Span}(\{\vec{r}_1, \vec{r}_2\}) = \text{Row}(B)$   
 $= \text{Row}(A)$

So,  $\{\vec{r}_1, \vec{r}_2\}$  is a basis for  $\text{Row}(A)$

I need  $c_1$  to be  
to "kill" the 1.  
The 0 here  
doesn't help.  
L4-22b: rein it  
idea

If  $c_1=0$ , the  
first line dies

B) Analyzing Span(S)

Ex Let  $S = \left\{ \underbrace{\begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ -15 \\ 3 \end{bmatrix}, \begin{bmatrix} 8 \\ 5 \\ 18 \end{bmatrix}}_{\text{in } \mathbb{R}^3} \right\}$

What is the dimension of  $\text{Span}(S)$ ?  
 ("How big?"), and

Find a basis for  $\text{Span}(S)$ .

Solution

Write the vectors as rows of A.

$$A = \begin{bmatrix} 1 & 4 & 3 \\ 2 & -1 & 4 \\ 3 & -15 & 3 \\ 8 & 5 & 18 \end{bmatrix}$$

$$\text{Span}(S) = \text{Row}(A)$$

find a basis

(row-echelon shape)

$$A \xrightarrow{\text{EROs}} B = \left[ \begin{array}{ccc|c} 1 & 4 & 3 & 0 \\ 0 & -9 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

↑ Grab the nonzero rows.

Many possible.

Basis for  $\text{Span}(S)$

$\leftrightarrow$  Basis for  $\text{Row}(A)$

$$= \{(1, 4, 3), (0, -9, -2)\} \left( \begin{array}{l} \rightarrow \text{efficient description} \\ \text{of } \text{Span}(S) \end{array} \right)$$

$\dim(\text{Span}(S))$

$$= \dim(\text{Row}(A))$$

$$= 2$$

The 4 given vectors span a 2-dim plane  
in  $\mathbb{R}^3$ .

### ⑥ Col(A)

Ex  $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 2 \\ 0 & 1 & 4 \\ 0 & 2 & 8 \end{bmatrix}$  (same as in ④)

$\uparrow \quad \uparrow \quad \uparrow$   
3 column vectors  
(in  $\mathbb{R}^4$ )

$\text{Col}(A)$ : column space of  $A$

$= \text{Span}(\text{column vectors})$   
'all linear combos'

is a subspace of  $\mathbb{R}^m$  (here,  $\mathbb{R}^4$ )

To find a basis for  $\text{Col}(A)$

Book does  
in HW  
Basis tends to  
be nice: leading Ps  
Book also  
shows

Method 1  $\text{Col}(A) = \underbrace{\text{Row}(A^T)}_{\text{find basis}}$

Method 2

$$A \xrightarrow{\text{EROS}} B \quad (\text{row-echelon shape})$$

Identify the columns of  $B$  with  $\textcircled{P}$ 's  
(pivot columns of  $B$ ).

Take the corresponding columns from  $A$ .  
(EROS may change the column space!)

In Ex

$$A = \left[ \begin{array}{cc|c} 1 & 2 & 3 \\ 2 & 3 & 2 \\ 0 & 1 & 4 \\ 0 & 2 & 8 \end{array} \right] \sim B = \left[ \begin{array}{ccc} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right]$$

Take cols.

1, 2 from  $A$

Pivot cols.

are cols. 1, 2

Basis for  $\text{Col}(A) =$

(works, because EROS preserve the dependency relationships among the columns)

$$\left\{ \left[ \begin{array}{c} 1 \\ 2 \\ 0 \\ 0 \end{array} \right], \left[ \begin{array}{c} 2 \\ 3 \\ 1 \\ 2 \end{array} \right] \right\}$$

$$\dim(\text{Col}(A)) = 2$$

$\text{Col}(A)$  is a 2-dim subspace of  $\mathbb{R}^4$

Why?  
complicated  
EROS  
preserve sets  
set of  $(A|B)$   
(i.e., dependency  
relationships)  
among the  
cols, but not  
col space.  
Ex:  $\vec{v}_1 + \vec{v}_2 = \vec{v}_3$

Take a look at  
 $B$ , there's no  
way we can  
get linear  
combs of cols  
that have  
non-0 entries  
in the 3rd, 4th  
components.  
But in  $A$

Ex  $\left( \begin{array}{c} 1 \\ 2 \\ 3 \\ 2 \end{array} \right)$

A bonus from Method 2 (though Method 1 often gives nicer bases with leading 0s):

$$\text{Also, } \text{Row}(A) \text{ basis} = \left\{ \begin{array}{l} \text{pivot rows from } B \\ (\text{nonzero}) \end{array} \right\} \\ = \{(1, 2, 3), (0, 1, 4)\}$$

Benefit of using Method 2  
can analyze  
Row(A), also  
Method 1 gives  
basis-leading 0s

Is that  
a coincidence? No!  
... or what?

$\dim(\text{Row}(A)) = 2$   
 $\text{Row}(A)$  is a 2-dim subspace of  $\mathbb{R}^3$ .

lowercase  
it's a #

### D rank(A)

$$\text{rank}(A) = \dim(\text{Row}(A)) = \dim(\text{Col}(A))$$

= # of PPs in any  
row-echelon shape of A

In Ex  $A \sim B = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$$\text{rank}(A) = 2$$

### E N(A)

$N(A) = \text{the nullspace of } A$   
 $= \{\vec{x} \mid A\vec{x} = \vec{0}\}$   
 $= \text{sol'n set (space) of } A\vec{x} = \vec{0}$   
 is a subspace of  $\mathbb{R}^n$ .

The nullity of  $A = \dim(N(A))$

Ex  $A = \begin{bmatrix} 1 & 0 & 3 & -1 \\ 0 & 2 & 4 & 0 \\ 2 & 1 & 8 & -2 \end{bmatrix}$

Find a basis for  $N(A)$   
and nullity  $(A)$

Solve  $A\vec{x} = \vec{0}$

$$\left[ \begin{array}{cccc|c} 1 & 0 & 3 & -1 & 0 \\ 0 & 2 & 4 & 0 & 0 \\ 2 & 1 & 8 & -2 & 0 \end{array} \right]$$

$\downarrow$  EROs Gauss-Jordan       $\uparrow$  never changes

$$\left[ \begin{array}{cccc|c} \textcircled{1} & 0 & 3 & -1 & 0 \\ 0 & \textcircled{1} & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right] \text{ RREF form}$$

$x_3, x_4$  free

$\rightarrow$  System

$$\begin{cases} x_1 + 3x_3 - x_4 = 0 \\ x_2 + 2x_3 = 0 \end{cases}$$

$\underbrace{\qquad}_{\text{free}}$

$$\begin{cases} x_1 = -3x_3 + x_4 \\ x_2 = -2x_3 \end{cases}$$

As you perform  
EROS, will  
this column  
of three  
ever  
change?

Worth reviewing,  
people tended  
not to do it  
my way on  
Mid 1

Parametrization

$$\text{Let } x_3 = t$$

$$x_4 = u$$

Sol'n Space  $N(A)$  in Parametric Form

$$\begin{cases} x_1 = -3t + u \\ x_2 = -2t \\ x_3 = t \\ x_4 = u \end{cases} \quad t, u \text{ are any real } \#s$$

→ Vector Notation

The only way we can get  $\vec{0}$  is if  $t, u$  are 0.

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = t \begin{bmatrix} -3 \\ -2 \\ 1 \\ 0 \end{bmatrix} + u \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad \begin{array}{l} \text{coeffs. of } t \\ \downarrow \\ \vec{v}_1 \end{array} \quad \begin{array}{l} \text{coeffs. of } u \\ \downarrow \\ \vec{v}_2 \end{array}$$

$\rightarrow \vec{v}_1, \vec{v}_2$   
 $\rightarrow$  are LI

So we're talking about all LGS

- the what of  $\vec{v}_1, \vec{v}_2$ ?

$t, u$  are any real #s

$$\text{So, } N(A) = \text{Span.}(\{\vec{v}_1, \vec{v}_2\}) \quad \text{subspace of } \mathbb{R}^4$$

A basis for  $N(A) = \{\vec{v}_1, \vec{v}_2\}$

$$= \left\{ \begin{bmatrix} -3 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$\text{nullity}(A) = 2$$

4.6.9

$$\text{F) } \underline{\text{rank}(A) + \text{nullity}(A) = n}$$

# pivot cols      # nonpivot cols      ↑ # cols in A

Old Ex

$$[A | \vec{b}]$$

$$\left[ \begin{array}{cccc|c} 1 & 0 & 3 & -1 & 0 \\ 0 & 2 & 4 & 0 & 0 \\ 2 & 1 & 8 & -2 & 0 \end{array} \right]$$

$$\sim \left[ \begin{array}{cccc|c} 1 & 0 & 3 & -1 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

↑      ↑      ↑      ↑  
 2 pivot cols      2 nonpivot cols.

$\Rightarrow \text{rank}(A)=2$        $\Rightarrow 2 \text{ free vars.}$   
 $(\Rightarrow N(A) \text{ is 2-dim.})$   
 $\Rightarrow \text{nullity}(A)=2$

You can have  
less than n.

Always  
 $(\# \text{ pivot cols.}) + (\# \text{ nonpivot cols.})$   
 $= \text{total } \# \text{ cols. in } A$   
 $\text{rank}(A) + \text{nullity}(A)$   
 $= n$

## ⑥ Systems (of Linear Eqs.)

Homogeneous:  $A\vec{x} = \vec{0}$

Sol'n set is a subspace, namely  $N(A)$ .

Nonhomogeneous:  $A\vec{x} = \vec{b}$  ( $\vec{b} \neq \vec{0}$ )

What's an easy way of seeing

Sol'n set is not a subspace.

$\vec{0}$  is not in it, since  $A\vec{0} = \vec{0}$

Same A as before

Ex Solve  $A\vec{x} = \vec{b}$  where

$$\left[ \begin{array}{cccc|c} 1 & 0 & 3 & -1 & 2 \\ 0 & 2 & 4 & 0 & 4 \\ 2 & 1 & 8 & -2 & 6 \end{array} \right]$$

same as before      ↗ will change

↓ Gauss-Jordan

$$\left[ \begin{array}{cccc|c} 1 & 0 & 3 & -1 & 2 \\ 0 & 1 & 2 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

Same RREF form as before      ↑ not a pivot col.  
→ consistent

doesn't have a leading non-0 entry

Sol'n Set in Parametric form

$$\begin{cases} x_1 = 2(-3t + u) \\ x_2 = 2(-2t) \\ x_3 = t \\ x_4 = u \end{cases} \quad \text{same as for } A\vec{x} = \vec{0}$$

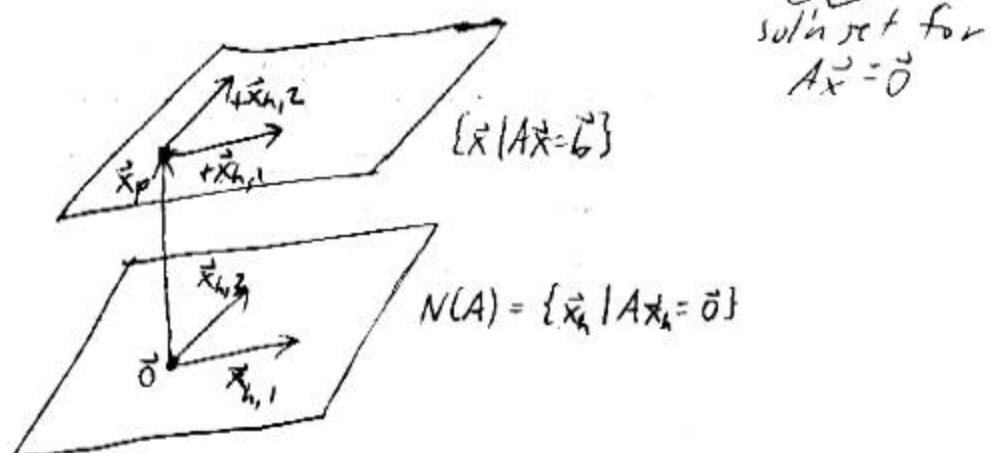
$t, u$  are any real #s

→ Vector Notation pushes  $N(A)$  away from  $\vec{0}$  (origin)

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \underbrace{\begin{bmatrix} 2 \\ 2 \\ 0 \\ 0 \end{bmatrix}}_{\text{constant terms}} + t \underbrace{\begin{bmatrix} -3 \\ -2 \\ 1 \\ 0 \end{bmatrix}}_{\vec{x}_p, \text{ a particular sol'n of } A\vec{x} = \vec{b}} + u \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}}_{\text{sweep out } N(A)}$$

$(t, u \text{ are any real #s})$   
 $= \{\vec{x}_h \mid A\vec{x}_h = \vec{0}\}$

If consistent,  
 Sol'n set for  $A\vec{x} = \vec{b}$  is "||" to  $N(A)$ .



(H)  $A\vec{x} = \vec{b}$  is consistent  $\Leftrightarrow \vec{b}$  is in  $\text{Col}(A)$

Why?

$$\begin{aligned} & A\vec{x} = \vec{b} \text{ consistent} \\ \Leftrightarrow & \left[ \vec{a}_1 \ \vec{a}_2 \ \dots \vec{a}_n \right] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \vec{b} \\ \xrightarrow{\text{In fact}} & \underbrace{x_1 \vec{a}_1 + x_2 \vec{a}_2 + \dots + x_n \vec{a}_n}_{\text{a linear combo of the cols. of } A} = \vec{b} \end{aligned}$$

$\Leftrightarrow \vec{b}$  is in  $\text{Span}(\{\text{cols. of } A\})$   
 $\text{Col}(A)$

(old)  
**Ex**

$$\left[ \begin{array}{cccc|c} 1 & 0 & 3 & -1 & 2 \\ 0 & 2 & 4 & 0 & 4 \\ 2 & 1 & 8 & -2 & 6 \\ \hline \vec{a}_1 & \vec{a}_2 & \vec{a}_3 & \vec{a}_4 & \vec{b} \end{array} \right]$$

One sol'n was  $\vec{x}_p = \begin{bmatrix} 2 \\ 2 \\ 0 \\ 0 \end{bmatrix}$

We're going to weight the cols of  $A$  by the coords of the sol'n.

$$\begin{aligned} & (x_1 \vec{a}_1 + x_2 \vec{a}_2 + x_3 \vec{a}_3 + x_4 \vec{a}_4) \\ & = 2 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} + 0 \begin{bmatrix} 3 \\ 4 \\ 8 \end{bmatrix} + 0 \begin{bmatrix} -1 \\ 0 \\ -2 \end{bmatrix} \\ & = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} \leftarrow \vec{b} !! \end{aligned}$$