4.2: VECTOR SPACES (VSs)

(A) What are they? Read

a set is just a group of objects

A set V is a vector space $\leftrightarrow$ the 10 properties (axioms) on p. 171 hold.

props. include comm., dist., assoc.

The members/elements of V are called vectors $(\vec{v}, \vec{w}, \text{etc.})$

If it's not obvious, you must define vector addition $(\vec{v}+\vec{w})$ and scalar multiplication $(c\vec{v})$

The set of scalars $(c, d, \text{etc.})$ is assumed to be $\mathbb{R}$ (the reals). Ch. 8: $\mathbb{C}$ (complex #s).

(B) Examples of VSs

m, n are fixed positive integers

in chaos and fractal theory, you deal w/ fractional dims.

Ex $\mathbb{R}^n$ $(\mathbb{R}^1, \mathbb{R}^2, \mathbb{R}^3, \text{etc.})$

$\vec{0}+\vec{0}$ may not $= \vec{0}$!

Don't mix $\mathbb{R}^2, \mathbb{R}^3$

$\begin{bmatrix}0\\0\end{bmatrix}+\begin{bmatrix}0\\0\\0\end{bmatrix}$ is undefined.

Ex $\underbrace{M_{3,2}}_{\text{"}V\text{"}}$ = set of all real 3×2 matrices

Vectors in V are 3×2 matrices!!

$$\vec{v} = \begin{bmatrix} 1 & -1 \\ 3 & 0 \\ 4 & \pi \end{bmatrix} \qquad \vec{w} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\vec{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

additive identity for V
For every $\vec{v}$ in V,
$\vec{v} + \vec{0} = \vec{v}$

$$-\vec{v} = \begin{bmatrix} -1 & 1 \\ -3 & 0 \\ -4 & -\pi \end{bmatrix}$$

additive inverse of $\vec{v}$
$\vec{v} + (-\vec{v}) = \vec{0}$

Define vector "+", scalar mult.

There are other, more exotic ways of defining

Use usual matrix ops.
(i.e., add corresp. entries for "+")

(i.e., to obtain $c\vec{v}$, mult. each entry of $\vec{v}$ by c)

All 10 props. on p.171 hold

(1) Closure under vector "+"

Your book likes $\vec{u}$, but we'll use $\vec{u}$ later. I don't like $\vec{u}$.

For every $\vec{v}$ and $\vec{w}$ in V, $\vec{v}+\vec{w}$ is in V.

We used these before

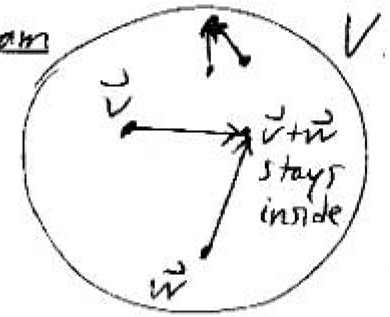
Ex

$$\vec{v}+\vec{w} = \begin{bmatrix} 1 & -1 \\ 3 & 0 \\ 4 & \pi \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 4 & 0 \\ 4 & \pi \end{bmatrix} \leftarrow \text{also in } V$$

Anytime you add two real 3×2 matrices, the sum is a real 3×2 matrix.

Venn Diagram



$\vec{v}$ could $= \vec{w}$

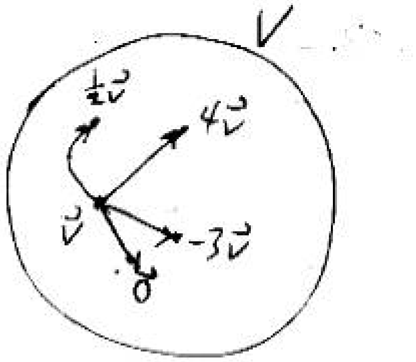
⑥ Closure under scalar mult.

Then... what?

For every $\vec{v}$ in V and every c in $\mathbb{R}$,
$c\vec{v}$ is in V.

Ex

$$\tfrac{1}{2}\vec{v} = \tfrac{1}{2}\begin{bmatrix} 1 & -1 \\ 3 & 0 \\ 4 & \pi \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{3}{2} & 0 \\ 2 & \frac{\pi}{2} \end{bmatrix}$$ also in V



②,③ Vector "+" is comm., assoc.

④ Additive identity

There is a $\vec{0}$ in V such that
$\vec{v} + \vec{0} = \vec{v}$ for every $\vec{v}$ in V.

$$\vec{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$ is in V

p.166: $\vec{v}$ = the neg. of $\vec{v}$ in a VS; p.174: $-\vec{v} = (-1)\vec{v}$
Me: Don't worry (V not a vec space) might not be $(-1)\vec{v}$ see Ex 8 $\vec{v} = (x_1, x_2)$ $(-1)\vec{v} = (-x_1, 0)$

⑤ Additive inverse

For every $\vec{v}$ in V, there is a vector $-\vec{v}$ in V
such that $\vec{v} + (-\vec{v}) = \vec{0}$.

⑦-⑩ Scalar props.

Exs of Vector Spaces

Ex $M_{m,n}$ = set of all real $m \times n$ matrices

Ex P = set of all polynomials in x, say

Use "standard ops." for poly. "+" and scalar mult.

$\vec{0} = 0$

The additive inverse of, say,
$\vec{v} = a_2x^2 + a_1x + a_0$ is
$-\vec{v} = -a_2x^2 - a_1x - a_0$

If $\vec{v} = 4x^3 - 2x$ and $\vec{w} = 2x^3 + 1$

then
$$\left.\begin{aligned} -\vec{v} &= -4x^3 + 2x \\ 3\vec{v} &= 3(4x^3 - 2x) = 12x^3 - 6x \\ \vec{v} + \vec{w} &= 6x^3 - 2x + 1 \end{aligned}\right\} \text{all in } P$$

(Quickly ✓ that the 10 props. hold.)

Ex P_2 = set of all polys in x of degree 2 or less
includes 0

(0 tech. has no degree)
Why "or less" we'll see in a moment

Ex P_n = " " degree n or less

Theorems for general VSs are powerful! They apply to many kinds of sets!

© Sets that are Not VSs

A set is a VS $\leftrightarrow$
all 10 props. hold for all cases

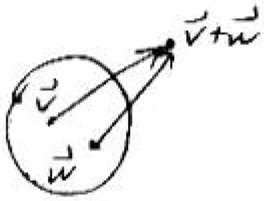
To show that a set is not a VS, you just need what?

One counterexample for any prop. $\Rightarrow$
not a VS.

Ex Set of polys in x of degree 2
w/standard ops.

assumed

If $\vec{v} = x^2 + x$
$\vec{w} = -x^2$
then $\vec{v} + \vec{w} = x \leftarrow$ not in set (deg=1)

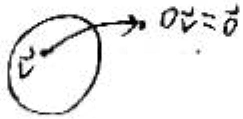


I know this set is not a VS. Why not?

So, ① Closure under "+" $\leftarrow$ FAILS !!
The set is not a VS.

Also, ④ fails : The set does not contain a $\vec{0}$.

⑥ fails : $0\vec{v} = \vec{0}$



Ex The set $\{(x,y) \mid x, y \text{ are whole \#s}\}$ ($0,1,2,\ldots$)
w/standard ops.

I'm including 0 "whole"

Can you think of a counterex? to one of the axioms? for Ex, (1,0) does not have a what in this set?

The additive inverse of $(1,0)$ is $\underbrace{(-1,0)}_{\text{but not in the set!}}$.

So, ⑤ fails. The set is not a VS.

Also, ⑥ fails: $\frac{1}{2}(1,0) = (\frac{1}{2},0) \leftarrow$ not in the set!
We do not have closure under scalar mult.