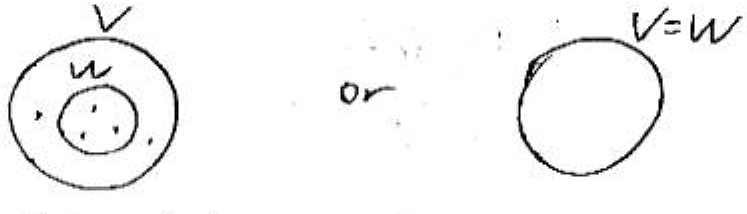
4.3: SUBSPACES OF VSs

(A) What is a Subspace?

Let V be a VS.

W is a subset of V $(W \subseteq V)$

$\Leftrightarrow$ all the vectors in W are also in V.



$W \subsetneq V$

$\uparrow$ is a ~~proper~~ subset of

$(W \subseteq V$, but $V \neq W)$

W is a subspace of V $\Leftrightarrow$

(1) $W \neq \phi$ (empty set)

(2) $W \subseteq V$

and (3) W is, itself, a VS

(with the same def'ns for vector "+" and scalar mult. used for V - usually assumed).

closure

except 1,4,5,6

+ id, inv.

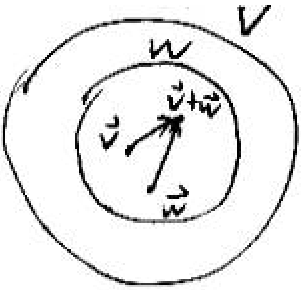
Just by virtue of being a subset

If $\overset{W \neq \emptyset,}{W \subseteq V}$, W <u>automatically inherits</u> most of the VS props. from V.

To show that W is a <u>subspace</u> of V, it's sufficient to show

① <u>Closure of W under vector "+"</u>

For every $\vec{v}, \vec{w}$ in $W \Rightarrow$
$\vec{v}+\vec{w}$ is also in W

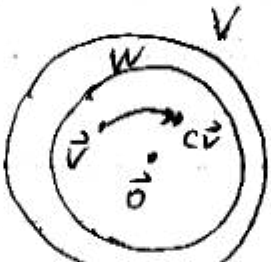


② <u>Closure of W under scalar mult.</u>

For every $\vec{v}$ in W and every real scalar $c \Rightarrow$
$c\vec{v}$ is also in W

what vector

In particular, $\vec{0}$ must be in W. $(c=0)$



auto inherits 6
you want to prove there 7
the 2 remaining props. come ~~d~~ there

Then, W will have all 10 VS props.

If V is any VS,

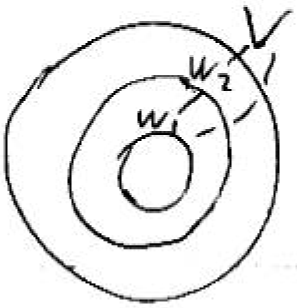
- its smallest subspace is $\{\vec{0}\}$ (<u>zero subspace</u>)
- its largest subspace is V
- any other subspace is a <u>proper (non trivial) subspace</u>

(B) <u>Relating Subspaces</u>

If W_1 is a subspace of W_2, and
if W_2 " " V

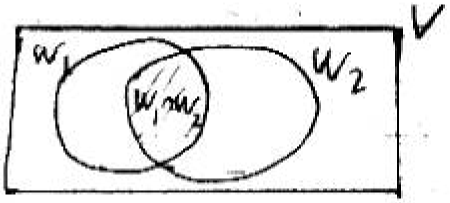
$\Rightarrow W_1$ " " V

< has the same prop'
$a < b,\ b < c \Rightarrow a < c$



transitivity

If W_1 and W_2 are subspaces of V
$\Rightarrow W_1 \cap W_2$ is also a subspace of V.
$\cap$ intersection



(C) Subspaces of $\mathbb{R}^1$

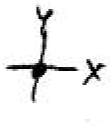
Only two — guess

Only $\{\vec{0}\}$, $\mathbb{R}^1$

No nontrivial subspaces

(D) Subspaces of $\mathbb{R}^2$

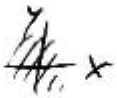
(1) $W = \{\vec{0}\}$
i.e., $\{(0,0)\}$



(2) (∞ many nontrivial subspaces)
W = set of all "points" on a straight line through (0,0)

technically, set of vectors that rep. points on the line

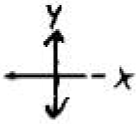
(3) $W = \mathbb{R}^2$



Prove (2)

Lines thru (0,0) take the form $y=mx$ (y-int is 0) What's the one line thru (0,0) not of this form?

Case 1



(slope "m" is undefined)

$W = \{(0,y) \mid y \text{ is a real } \#\}$

$W \neq \emptyset$, $W \subseteq \mathbb{R}^2$

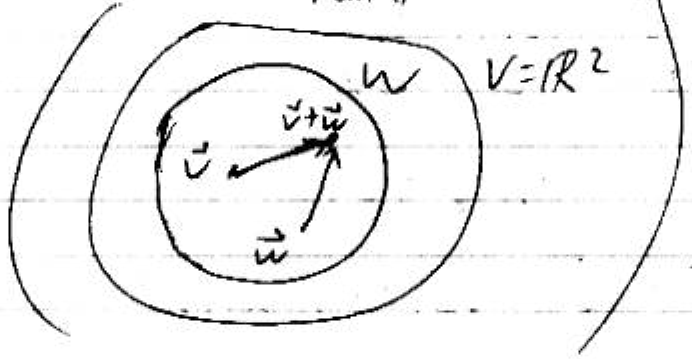
Let $\vec{v}, \vec{w}$ be arbitrary vectors in W.

$\vec{v} = (0, y_1)$
$\vec{w} = (0, y_2)$

y_1, y_2 are real #s

Prove: W is closed under vector "+"

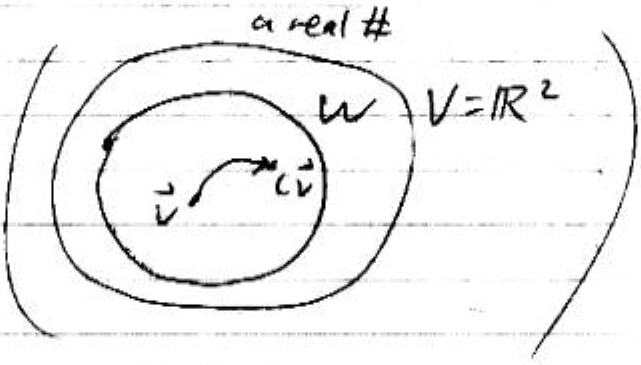
$\vec{v} + \vec{w} = (0, \underbrace{y_1 + y_2}_{\text{real \#}})$ is in W



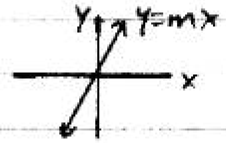
Prove: W is closed under scalar mult.

If c is any real scalar,

$c\vec{v} = (0, \underbrace{cy_1}_{\text{a real \#}})$ is in W



Case 2



On line, $x = t$

$\Rightarrow y = mt$

$W = \{(\underbrace{t}_{\text{a real \#}}, \underbrace{mt}_{\text{m(some \#)}}) \mid t \text{ is a real \#}\}$

$W \neq \phi$, $W \subseteq \mathbb{R}^2$

Let $\vec{v}, \vec{w}$ be arbitrary vectors in W.

$$\vec{v} = (t_1, mt_1)$$
$$\vec{w} = (t_2, mt_2)$$

t_1, t_2 are real #s

$$\vec{v} + \vec{w} = (t_1 + t_2, mt_1 + mt_2)$$

$$= (\underbrace{t_1 + t_2}_{\text{a real \#}}, \underbrace{m(t_1 + t_2)}_{\text{m(same real \#)}})$$

or let $t_3 = t_1 + t_2$

is in W ✓

If c is any real scalar, $\vec{v} = (t_1, mt_1) \rightarrow$

we need form (#, m·#)

$$c\vec{v} = (ct_1, cmt_1)$$

$$= (\underbrace{ct_1}_{\text{a real \#}}, m\underbrace{(ct_1)}_{\text{same real \#}})$$

a real # → same real #

is in W ✓

quod erat demonstrandum

end of proof
that which was
to be proven
Klatu verata nikto

QED

Note: Can test $\vec{v} + c\vec{w}$ to prove closure props. simultaneously.

$c = 1 \rightarrow +$
$\vec{v} = \vec{0} \rightarrow$ s.m.

(E) Subspaces of $\mathbb{R}^3$

	Later: "dim" (dimension)
(1) $W = \{\vec{0}\}$	0
(2) (∞ many) W = set of all "points" on a straight line through $(0,0,0)$	1
(3) (∞ many) W = " , plane through $(0,0,0)$	2
(4) $W = \mathbb{R}^3$	3

WARNING: Is $\mathbb{R}^2$ a subspace of $\mathbb{R}^3$?

NO $\mathbb{R}^2 \not\subseteq \mathbb{R}^3$

It won't work if you try to mix these - vector + is often/typically undefined

$\mathbb{R}^2$ $\cdot\begin{bmatrix}0\\0\end{bmatrix}$ $\quad$ $\cdot\begin{bmatrix}0\\0\\0\end{bmatrix}$ $\mathbb{R}^3$ Don't mix!

(F) Linear Eqs.

The graph of any homog. linear eq. in n vars. represents a subspace of $\mathbb{R}^n$.

Ex $2x - y + 3z = 0$

Graph: subspace of $\mathbb{R}^3$ (plane through $\vec{0}$)

generaliz'n

The solution set of $A\vec{x}=\vec{0}$ is a subspace of $\mathbb{R}^n$. (HW - #26)

<u>Ex</u> The sol'n set of

$$\begin{cases} x_1 \qquad\qquad + x_4 = 0 \\ \quad x_2 - x_3 + 2x_4 = 0 \end{cases}$$

is a subspace of $\mathbb{R}^4$.

⑥ Examples

Read all Exs, except Ex 5.

<u>Ex</u> $W = \left\{ \begin{bmatrix} 1 & a \\ b & c \end{bmatrix} \middle| \ a,b,c \text{ are real \#s} \right\}$

Is W a subspace of $M_{2,2}$?

Easy ans.

NO $\quad \vec{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is not in W

The poor $\vec{0}$

(Just need one counterexample!)

Is W closed under vector +?

<u>or</u> $\begin{bmatrix} 1 & a_1 \\ b_1 & c_1 \end{bmatrix} + \begin{bmatrix} 1 & a_2 \\ b_2 & c_2 \end{bmatrix} = \begin{bmatrix} 2 & - \\ \sim & \end{bmatrix}$ not in W

<u>or</u> $\pi \begin{bmatrix} 1 & a_1 \\ b_1 & c_1 \end{bmatrix} = \begin{bmatrix} \pi & \sim \\ \sim & \end{bmatrix}$ not in W