4.4: SPANNING SETS + LINEAR INDEPENDENCE

Read

Ⓐ Linear Combinations of Vectors

Let V be a VS.
Let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ be vectors in V.

Then, $\vec{w}$ is a **linear combination** of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ $\leftrightarrow$

$\vec{w}$ can be written as

real # times $\vec{v}_1 \ldots$

$$\vec{w} = c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_k\vec{v}_k$$

where the "c_i"s are scalars.

Ex Some linear combos of $\overbrace{\vec{v}_1, \vec{v}_2, \vec{v}_3}^{\text{in } \mathbb{R}^{10}\text{, say}}$:

$3\vec{v}_1 - 4\vec{v}_2 + \vec{v}_3$	$c_1 = 3,$	$c_2 = -4,$	$c_3 = 1$
$\pi\vec{v}_3$	0	0	π
what special vector $\vec{0}$	0	0	0

Ex Let $\vec{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ ↖
$\vec{v}_2 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$ →

Express $\vec{w} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$ as a linear combo of $\vec{v}_1, \vec{v}_2$.

Idea

Need c_1, c_2 such that

$$c_1 \vec{v}_1 + c_2 \vec{v}_2 = \vec{w}$$

Trust me!

$$\underbrace{\begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix}}_{\text{"}A\text{"}} \underbrace{\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}}_{\text{"}\vec{c}\text{"}} \stackrel{?}{=} \vec{w}$$

weights for cols. of A

stack the v vectors as columns

not block mult. You can informally look at it like this $v_1 \, c_1 + v_2 \, c_2$

Work

Solve $A\vec{c} = \vec{w}$

$$\left[\begin{array}{cc|c} \vec{v}_1 & \vec{v}_2 & \vec{w} \end{array}\right]$$

$$\left[\begin{array}{cc|c} -1 & 2 & 4 \\ 1 & 0 & 2 \end{array}\right]$$

$\downarrow$ Gauss-Jordan

$$\left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 3 \end{array}\right]$$

$c_1 = 2$
$c_2 = 3$

$\vec{w} = c_1\vec{v}_1 + c_2\vec{v}_2$

$\vec{w} = 2\vec{v}_1 + 3\vec{v}_2$ Can ✓: $\begin{bmatrix}4\\2\end{bmatrix} = \begin{bmatrix}-2\\2\end{bmatrix} + \begin{bmatrix}6\\0\end{bmatrix}$

<u>Picture</u>

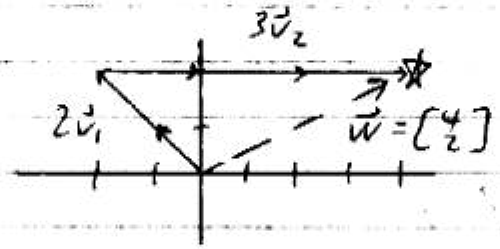
Al Bundy's dodge
Treasure: 3 tickets

$\vec{w} = 2\begin{bmatrix}-1\\1\end{bmatrix} + 3\begin{bmatrix}2\\0\end{bmatrix}$

(I'll write later)

2 paces this way, where ↑ is a pace

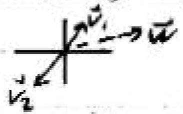
Only way if no backtracking no permuting



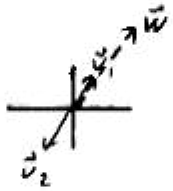
"treasure map"

like having a car that only goes forward + backward

If $A\vec{c} = \vec{w}$ is inconsistent, then $\vec{w}$ <u>can't</u> be expressed as a linear combo of the "v_i"s. <u>Ex</u>



If $A\vec{c} = \vec{w}$ has ∞ many sol'ns, then ∞ many linear combos work <u>Ex</u>



Can do 1, 3a

(B) Span(S)

Let S (be the set of vectors) $\underbrace{\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\}}_{\text{in } V}$

A robot programmed to walk in these directions.

"c_i"s can be any seq. of real #s

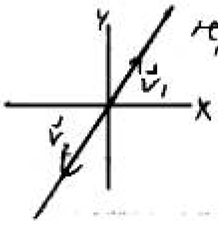
Then, Span(S) = the set of all linear combos of "$\vec{v}_i$"s ("what we can reach")

$= \{c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_k\vec{v}_k \mid \text{"}c_i\text{"s are real \#s}\}$

is a subspace of V

Ex

What points can we reach?



represents Span($\{\vec{v}_1, \vec{v}_2\}$)
- subspace of $\mathbb{R}^2$

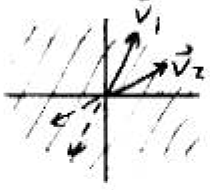
If Span(S) = V, then

"S spans V"
"S is a spanning set of V"

Any vector in V can be written as a linear combo of "$\vec{v}_i$"s.

Ex

In a sense, we can hit every pt in the plane, using these direction vectors



$\text{Span}(\underbrace{\{\vec{v}_1, \vec{v}_2\}}_{S}) = \mathbb{R}^2$

S spans $\mathbb{R}^2$

"c_i"s can be < 0.

"HW" Ex

$$S = \left\{ \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 6 \\ 2 \\ 8 \end{bmatrix} \right\}$$

Does S span $\mathbb{R}^3$?
If not, what does S span?

$$\text{Let } A = \begin{bmatrix} 2 & 2 & 6 \\ 0 & 1 & 2 \\ 2 & 3 & 8 \end{bmatrix}$$

$$\underset{\sim}{\text{EROs}} \begin{bmatrix} 2 & 2 & 6 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$ row-echelon shape (leading 1s unnecessary)

2 pivot positions (PPs) (not in book), not 3

S does not span $\mathbb{R}^3$.

S spans a 2-dim. subspace (plane) in $\mathbb{R}^3$.

Why? - Later

$$\boxed{\begin{array}{l} \text{The cols. of } A \text{ span } \mathbb{R}^m \Leftrightarrow \\ \text{each row has a PP} \\ \text{(LI: each column)} \end{array}}$$

Up to 15

Ⓒ Beyond $\mathbb{R}^n$

How many entries

Under the usual ops., there is a "natural" correspondence between $M_{m,n}$ and $\mathbb{R}^{mn}$ and between P_n and $\mathbb{R}^{n+1}$

Ex $M_{2,3}$ = set of real 2×3 matrices

Treat $\vec{v} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ in $M_{2,3}$

as, say, $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{bmatrix}$ in $\mathbb{R}^6$.

Be consistent If you have another matrix that's 2×3, Write the #s in the same order.

Ex P_2 = set of all polys in x of degree 2 or less (includes 0)

If you +, − poly's, or you · by real #s → all the action is with the coeffs.

Treat $\vec{v} = ax^2 + bx + c$ in P_2

as, say, $\begin{bmatrix} a \\ b \\ c \end{bmatrix} \begin{pmatrix} \leftarrow \text{coeff. of } x^2 \\ \leftarrow \text{coeff. of } x \\ \leftarrow \text{constant term} \end{pmatrix}$ in $\mathbb{R}^3$.

⓪ Linear Independence

The set $S = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\}$ is a <u>linearly independent (LI)</u> set $\leftrightarrow$

If you take LCs of these vecs, the only way you can get $\vec{0}$ is if coeffs are also 0.

$c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_k\vec{v}_k = \vec{0}$ has only the trivial sol'n $c_1 = 0, c_2 = 0, \dots, c_k = 0$

i.e., (if $V = \mathbb{R}^m$)

"Trust me" step

$$\underbrace{\begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_k \end{bmatrix}}_{\substack{A \\ (m \times k)}} \underbrace{\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{bmatrix}}_{\substack{\vec{c} \\ (\text{weights})}} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}}_{\vec{0}}$$

has only the trivial sol'n $\vec{c} = \vec{0}$.

Otherwise, S is <u>linearly dependent (LD)</u>.

<u>Idea</u> (in $\mathbb{R}^m$)

Each vector in the set is "valuable"

<u>Each</u> vector in a LI set is needed to "reach new points" that couldn't be reached, otherwise. "Valuable"

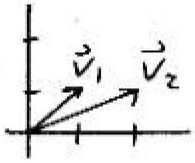
<u>Ex</u> $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$ is a LI set.

$\vec{v}_1 \quad \vec{v}_2$

4 Reasons:

$\vec{v}_1$ helps us reach points that $\vec{v}_2$ alone couldn't, and v-v
if you have a car
if you can go ... $\vec{v}_2$, better.

(#1)



Each vector is "valuable" for reaching new points.

related to Reason 1

(not ∥) If one were a scalar mult. of the other, then one would be redundant in terms of helping us reach new pts.

(#2) A set of <u>two</u> vectors in V is LI $\Leftrightarrow$ neither is a scalar multiple of the other. Not ∥.

Method works for a set of ? vectors in $\mathbb{R}^n$.

(#3) (General Method)

$A\vec{c} = \vec{0}$ has only the trivial sol'n, $\vec{c} = \vec{0}$.

$$\left[\underbrace{\vec{v}_1 \quad \vec{v}_2}_{A} \;\middle|\; \vec{0} \right]$$

$$\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 1 & 1 & 0 \end{array} \right]$$

$$\sim \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & -1 & 0 \end{array} \right]$$ row-echelon shape

each col. has PP — ignore

To get to row-echelon form, just rescale row 2 by (-1), but let's look at this. I don't care if -1 or 3, ... as long as ≠0

A may not = I.
skinny

that means
system has no...

$\Rightarrow$ no free vars

$\Rightarrow$ $A\vec{c}=\vec{0}$ has only trivial sol'n

$\rightarrow$ Columns of A $(\vec{v}_1, \vec{v}_2)$ form a LI set.

i.e., $c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{0}$

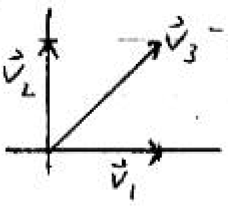
only 0s will work

(#4) ✓ $\det(A) \neq 0$, if A is square.

Ex $\left\{ \underset{\vec{v}_1}{\begin{bmatrix}1\\0\end{bmatrix}}, \underset{\vec{v}_2}{\begin{bmatrix}0\\1\end{bmatrix}}, \underset{\vec{v}_3}{\begin{bmatrix}1\\1\end{bmatrix}} \right\}$ is a LD set. (even though no vector is a multiple of any other!)

4 Reasons:

(#1)



(#2)

We can see: "dependency relation"

$\vec{v}_1 + \vec{v}_2 = \vec{v}_3$

$\vec{v}_1 + \vec{v}_2 - \vec{v}_3 = \vec{0}$

one vec can be expressed as a linear combo of others

means $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$ has a nontrivial sol'n

$c_1 = 1,\ c_2 = 1,\ c_3 = -1$

∞ many others
2,2,-2

#3 (General Method)

$A\vec{c}=\vec{0}$ has nontrivial sol'ns.

$$\Big[\underbrace{\vec{v}_1 \quad \vec{v}_2 \quad \vec{v}_3}_{A}\ \Big|\ \vec{0}\Big]$$

already in RREF form!

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

2, not 3 PPs free var. ignore

Related

#4

A set of k vectors in $\mathbb{R}^m$ where $k>m$ must be LD.

#vecs > length or dim

What kind of coeff matrix

$$\begin{array}{cc} & \begin{array}{cc} 1 & k \end{array} \\ \begin{array}{c} 1 \\ m \end{array} & \Big[\ A_{m\times k} - \text{wide}\ \Big|\ \vec{0}\ \Big] \end{array}$$

$\Rightarrow$ free vars.

Is this LI or LD?

Ex $\left\{\begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}2\\3\end{bmatrix}\right\}$ is LD

Kato Kaelin of vectors

$\vec{0}$ never valuable. Any set with $\vec{0}$ is LD.

$$c_1\begin{bmatrix}0\\0\end{bmatrix} + c_2\begin{bmatrix}2\\3\end{bmatrix} = \vec{0}$$

c_2: 0

any real # (c_1 - free var)