4.5: BASIS and DIMENSION

Ⓐ What is a Basis?

Let V be a VS.
Let $S = \underbrace{\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m\}}_{\text{in } V}$

S is a basis for $V \leftrightarrow$

① S spans V, and
② S is LI

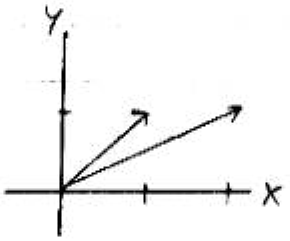
Idea

A basis must have enough vectors to span V, but all of them must be "valuable." It efficiently describes V.

There are no superfluous ones.

(If I ask for a VS, give me a basis, and I take all LCs of the basis vectors to sweep out the VS.)

Ex $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$ is a basis for $\mathbb{R}^2$



$A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$

Each row has a PP col.

→ cols of A span $\mathbb{R}^2$
→ LI

So, basis.

A basis for $\mathbb{R}^m$ must have exactly m vectors.

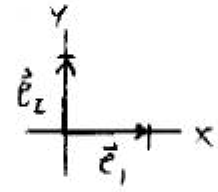
↗ dimension of $\mathbb{R}^m$

Sometimes if we have a VS, there is a very natural choice for a basis.

B) Standard Basis for ...

$\mathbb{R}^2$

$$\left\{ \underset{\vec{e}_1}{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}, \underset{\vec{e}_2}{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} \right\}$$



$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{array}{l} \rightarrow \text{cols} \\ \rightarrow \text{span } \mathbb{R}^2 \end{array}$$

$\downarrow$ cols LI

Building up I_m

$\mathbb{R}^m$

$$\left\{ \underset{\vec{e}_1}{\begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}}, \underset{\vec{e}_2}{\begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}}, \cdots, \underset{\vec{e}_m}{\begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}} \right\}$$

$A = I_m$

$M_{m,n}$

Ex $M_{2,3}$ (has dim = 6)

not ordered
could do $\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ 2nd
Later, coords wrt basis.
Examine ideas w/ E

$$\left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}$$

P_2: How many coeffs.

P_n: polys (in x) of degree $\le n$, incl. 0

Ex P_2: ax^2+bx+c $\{1, x, x^2\}$

$\{1, x, x^2, \ldots, x^n\}$

Up to 4

$\dim(P_n) = n+1$

Skip to C

E <u>A vector has unique coords relative to an ordered basis.</u>

Careful: coords require ordering

This is the order we're sticking with.

Let $B = \{\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_m\}$ be an ordered basis for V.

Each vector in V can (span) be written in <u>exactly one way</u> (LI) as a linear combo of "$\vec{b}_i$"s.

i.e., (in $\mathbb{R}^m$)

$A\vec{c} = \vec{w}$ has a <u>unique</u> sol'n for every $\vec{w}$ in $\mathbb{R}^m$.

A: $[\vec{b}_1\ \vec{b}_2 \cdots \vec{b}_m]$ $(m \times m)$

$\vec{c}$: <u>coords. of $\vec{w}$ rel. to B</u>

Why we need C before D

<u>When is B a basis for $\mathbb{R}^m$?</u>

Use Invert. Matrix Thm.

$\Leftrightarrow$ A is invertible

$\Leftrightarrow$ $A \sim I_m$ $\Leftrightarrow$ $\begin{bmatrix} P & \\ 0 & P \end{bmatrix}$

$\Leftrightarrow$ $\det(A) \ne 0$

etc.

Full set of (P)s down main diag

<u>Ex</u> Let $B = \left\{ \underset{\vec{b}_1}{\begin{bmatrix} -1 \\ 1 \end{bmatrix}}, \underset{\vec{b}_2}{\begin{bmatrix} 2 \\ 0 \end{bmatrix}} \right\}$.

ⓐ Show that B is a basis for $\mathbb{R}^2$.

① 2 vectors, not $\parallel$ $\rightarrow$ LI

2 vectors in $\mathbb{R}^2$ $\rightarrow$ span

$\Big\}$ basis

② $$\underset{A}{\begin{bmatrix} -1 & 2 \\ 1 & 0 \end{bmatrix}} \xleftarrow{} \text{Square} \qquad \begin{bmatrix} -1 & 2 \\ 1 & 0 \end{bmatrix} \sim \begin{bmatrix} -1 & 2 \\ 0 & 2 \end{bmatrix}$$

2 PPs

$$\left(\sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right)$$

or ③ $$\begin{vmatrix} -1 & 2 \\ 1 & 0 \end{vmatrix} = -2 \neq 0$$

ⓑ Write $\vec{w} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$ as a linear combo of $\vec{b}_1, \vec{b}_2$. (4.4 Ⓐ)

Solve $A\vec{c} = \vec{w}$

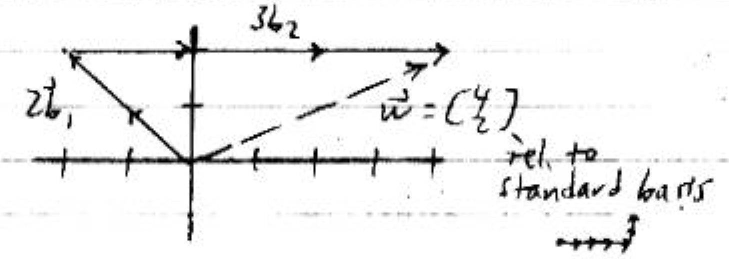
$$\left[\begin{array}{cc|c} -1 & 2 & 4 \\ 1 & 0 & 2 \end{array}\right]$$

$\downarrow$ Gauss-Jordan

$$\left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 3 \end{array}\right]$$

coords of $\vec{w}$ rel. to new basis

$\vec{w} = 2\vec{b}_1 + 3\vec{b}_2$ ("unique basis rep.")



In principle, you can talk about backtracking or zig-zagging, but after you combine like terms, you get this.

This is essentially the only way to get $\vec{w}$. (after combining like terms) "smooths out" any backtracking or zig-zagging

⓪ dim(V)

$=$ dimension of V (VS)

$=$ # of vectors in any basis for V

If $V = \{\vec{0}\}$, then $\dim(V) = 0$.

Me: all VSs have a convers. to $\mathbb{R}$ (or $\mathbb{C}$) thru coeffs of linear combos rel to a basis. (though ops may vary?!)

Exs $\dim(\mathbb{R}^n) = n$
$\dim(M_{m,n}) = mn$
$\dim(P_n) = n+1$

① A set of k vectors ($k > \dim(V)$; "too many") can't be LI.

#vecs > dim(V)

Idea Not all of them can be valuable.

<u>Ex</u> $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ is clearly LD 

$3 > \underset{\uparrow \dim(\mathbb{R}^2)}{2}$ or $\underset{\text{"fat"}}{\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}}$ can't have a PP in each col.

#vecs < dim of V's

Why can't they form a basis?

② A set of k vectors ($k < \dim(V)$; "too few") can't span V.

<u>Ex</u> $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ clearly doesn't span $\mathbb{R}^3$ 

$2 < \underset{\uparrow \dim(\mathbb{R}^3)}{3}$ or $\underset{\text{"skinny"}}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}}$ can't have a PP in each row

③ A set of k vectors ($k = \dim(V)$) <u>could</u> be a basis for V.

If you have a square matrix, each row has a PP ↔ col

Need: ① They span V ② They're LI — If you have one, you have both if k vectors, $k = \dim(V)$.

<u>Ex</u> (in $\mathbb{R}^n$) $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is a basis $\Leftrightarrow$

$$\underset{\text{square}}{\begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{bmatrix}} \overset{\text{EROs}}{\sim} \underset{\substack{\text{Each col, row} \\ \text{has a PP.}}}{\begin{bmatrix} PP & & & \\ & PP & & \\ & 0 & \ddots & \\ & & & PP \end{bmatrix}} \sim I$$

Up to 29

① "HW" Ex

Find a basis for $U_{2,2}$ = VS of all upper triangular 2×2 matrices.

Generic form: $\left\{\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \middle| \; a,b,c: \text{any real \#s}\right\}$

What's a basis? –

$$= \left\{\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & c \end{bmatrix} \middle| \; '' \quad '' \right\}$$

$$= \left\{ \underbrace{a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}}_{\text{3 basis vectors}} \middle| \; '' \quad '' \right\}$$

LI ✓ (only $a=b=c=0 \Rightarrow 0_{2\times2}$)

when I take all poss LCs

Span $U_{2,2}$ ✓

→ Go back to E