4.6: RANK and SYSTEMS

A - $m \times n$

(A) Row(A)

Ex

$$\underset{(4\times 3)}{A} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 2 \\ 0 & 1 & 4 \\ 0 & 2 & 8 \end{bmatrix}$$ ← 4 row vectors (in $\mathbb{R}^3$)

Row(A) = row space of A

= Span(row vectors)
all linear combos

is a subspace of $\mathbb{R}^n$ (here, $\mathbb{R}^3$)

To find a basis for Row(A)

$A \xrightarrow{\text{EROs}} B$ (any row-echelon shape)
Take the nonzero row vectors from B. "Pivot rows"

In Ex

$$A \sim B = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Basis for Row(A) = $\{(1,2,3), (0,1,4)\}$

$\dim(\text{Row}(A)) = 2$

Why?

If you switch rows...

If $A \overset{\text{EROs}}{\sim} B$, then $Row(A) = Row(B)$

EROs do not disturb the row space.

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{array}{l} \leftarrow \vec{r}_1 \\ \leftarrow \vec{r}_2 \\ \\ \\ \end{array}$$

The easy answer here: neither is a multiple of the other

Why is $\{\vec{r}_1, \vec{r}_2\}$ guaranteed to be LI?

2 vectors (not $\parallel$)

or

I need c_1 to be 0 to "kill" the 1! The 0 here doesn't help. L4-22b: revisit idea

$$\begin{array}{rl} c_1 & [\,1 \quad 2 \quad 3\,] \\ + \; c_2 & [\,0 \quad 1 \quad 4\,] \\ \hline = & [\,0 \quad 0 \quad 0\,] \end{array}$$

If $c_1 = 0$, the first line dies

$\downarrow$ So, $c_1 = 0$ $\quad$ $\downarrow$ So, $c_2 = 0$

By definition, $Span(\{\vec{r}_1, \vec{r}_2\}) = Row(B)$
$= Row(A)$

So, $\{\vec{r}_1, \vec{r}_2\}$ is a basis for $Row(A)$

Ⓑ Analyzing Span(S)

Ex Let $S = \underbrace{\left\{ \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ -15 \\ 3 \end{bmatrix}, \begin{bmatrix} 8 \\ 5 \\ 18 \end{bmatrix} \right\}}_{\text{in } \mathbb{R}^3}$

What is the dimension of Span(S)?
("How big?"), and

Find a basis for Span(S).

Solution

Write the vectors as rows of A.

$$A = \begin{bmatrix} 1 & 4 & 3 \\ 2 & -1 & 4 \\ 3 & -15 & 3 \\ 8 & 5 & 18 \end{bmatrix}$$

$$\text{Span}(S) = \underbrace{\text{Row}(A)}_{\text{find a basis}}$$

(row-echelon shape)

$$A \overset{\text{EROs}}{\sim} B = \begin{bmatrix} 1 & 4 & 3 \\ 0 & -9 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$ Grab the nonzero rows.

Many possibs.

Basis for $\mathrm{Span}(S)$

$\leftrightarrow$ Basis for $\mathrm{Row}(A)$

$= \{(1,4,3),\ (0,-9,-2)\}$ ($\rightarrow$ efficient description of $\mathrm{Span}(S)$)

$\dim(\mathrm{Span}(S))$

$= \dim(\mathrm{Row}(A))$

$= 2$

The 4 given vectors span a 2-dim plane in $\mathbb{R}^3$.

(c) $\mathrm{Col}(A)$

Ex

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 2 \\ 0 & 1 & 4 \\ 0 & 2 & 8 \end{bmatrix}$$ (same as in (A))

3 column vectors (in $\mathbb{R}^4$)

$\mathrm{Col}(A) =$ column space of A

$= \mathrm{Span}(\text{column vectors})$

all linear combos

is a subspace of $\mathbb{R}^m$ (here, $\mathbb{R}^4$)

To find a basis for Col(A)

Book does in HW
Basis tends to be nice: leading 1s
Book also shows

Method 1 $\mathrm{Col}(A) = \underbrace{\mathrm{Row}(A^T)}_{\text{find basis}}$

Method 2

$$A \xrightarrow{\text{EROs}} B \quad \text{(row-echelon shape)}$$

Identify the columns of B with PPs (pivot columns of B).

Take the corresponding columns from A. (EROs may change the column space!)

Why? complicated

EROs preserve sol'n set of (A|B) (i.e., dependency relationships among the cols!), but not col space.
3rd Ex: $\vec{v}_1 + \vec{v}_2 = \vec{v}_3$
Take a look at B, there's no way we can get linear combos of cols that have non-0 entries in the 3rd, 4th components.
But in A
Ex $\begin{pmatrix} 2 \\ 3 \\ 1 \\ 2 \end{pmatrix}$

In Ex

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 2 \\ 0 & 1 & 4 \\ 0 & 2 & 8 \end{bmatrix} \sim B = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Take cols. 1,2 from A ← Pivot cols. are cols. 1,2

(works, because EROs preserve the dependency relationships among the columns)

Basis for $\mathrm{Col}(A) =$

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 1 \\ 2 \end{bmatrix} \right\}$$

$\dim(\mathrm{Col}(A)) = 2$

$\mathrm{Col}(A)$ is a 2-dim subspace of $\mathbb{R}^4$

A bonus from Method 2 (though Method 1 often gives nicer bases with leading 0s):

Benefit of using Method 2 can analyze Row(A), also Method 1 - nicer basis - leading 0s

Also, Row(A) basis = {pivot rows from B}
(nonzero)

$= \{(1,2,3), (0,1,4)\}$

Is that a coincidence? No!
... of $\mathbb{R}^{\text{what}}$?

$\dim(\text{Row}(A)) = 2$

Row(A) is a 2-dim subspace of $\mathbb{R}^3$.

lowercase - it's a #

D rank(A)

$\text{rank}(A) = \dim(\text{Row}(A)) = \dim(\text{Col}(A))$

$= \#$ of PPs in any row-echelon shape of A

In Ex. $\left(A \sim B = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right)$

$\text{rank}(A) = 2$

E N(A)

$N(A)$ = the nullspace of A

$= \{\vec{x} \mid A\vec{x} = \vec{0}\}$

= sol'n set (space) of $A\vec{x} = \vec{0}$

is a subspace of $\mathbb{R}^n$.

The nullity of $A = \dim(N(A))$

<u>Ex</u> $A = \begin{bmatrix} 1 & 0 & 3 & -1 \\ 0 & 2 & 4 & 0 \\ 2 & 1 & 8 & -2 \end{bmatrix}$

Find a basis for $N(A)$
<u>and</u> nullity(A)

Solve $A\vec{x} = \vec{0}$

As you perform EROs, will this column of zeros ever change?

$$\left[\begin{array}{cccc|c} 1 & 0 & 3 & -1 & 0 \\ 0 & 2 & 4 & 0 & 0 \\ 2 & 1 & 8 & -2 & 0 \end{array}\right]$$

never changes

$\Big\downarrow$ EROs Gauss-Jordan

$$\left[\begin{array}{cccc|c} 1 & 0 & 3 & -1 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$ RRE form

x_3, x_4 free

Worth reviewing; people tended not to do it my way on Mid 1

<u>→ System</u>

$$\begin{cases} x_1 + 3x_3 - x_4 = 0 \\ x_2 + 2x_3 = 0 \end{cases}$$

free

$$\begin{cases} x_1 = -3x_3 + x_4 \\ x_2 = -2x_3 \end{cases}$$

Parametrization

Let $x_3 = t$
$x_4 = u$

Sol'n Space N(A) in Parametric Form

$$\begin{cases} x_1 = -3t + u \\ x_2 = -2t \\ x_3 = t \\ x_4 = u \end{cases}$$

t,u are any real #s

→ Vector Notation

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = t \underbrace{\overbrace{\begin{bmatrix} -3 \\ -2 \\ 1 \\ 0 \end{bmatrix}}^{\text{coeffs. of } t}}_{\vec{v}_1} + u \underbrace{\overbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}}^{\text{coeffs. of } u}}_{\vec{v}_2} \rightarrow \vec{v}_1, \vec{v}_2 \text{ are LI}$$

The only way we can get $\vec{0}$ is if t,u are 0.

t,u are any real #s

So we're talking about all LCs

So, $N(A) = \underbrace{\text{Span}(\{\vec{v}_1, \vec{v}_2\})}_{\text{subspace of } \mathbb{R}^4}$

= the what of $\vec{v}_1, \vec{v}_2$?

A basis for $N(A) = \{\vec{v}_1, \vec{v}_2\}$

$$= \left\{ \begin{bmatrix} -3 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

nullity(A) = 2

Ⓕ $\underline{\text{rank}(A) + \text{nullity}(A) = n}$

$\underbrace{\text{rank}(A)}_{\#\text{pivot cols}} + \underbrace{\text{nullity}(A)}_{\#\text{nonpivot cols}} = \underbrace{n}_{\#\text{cols in } A}$

Old Ex

$[A \mid \vec{0}]$

$$\left[\begin{array}{cccc|c} 1 & 0 & 3 & -1 & 0 \\ 0 & 2 & 4 & 0 & 0 \\ 2 & 1 & 8 & -2 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & 3 & -1 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

You can have till #5.

2 pivot cols $\Rightarrow$ rank$(A) = 2$

2 nonpivot cols. $\Rightarrow$ 2 free vars. ($\Rightarrow N(A)$ is 2-dim.) $\Rightarrow$ nullity$(A) = 2$

Always

(# pivot cols.) + (# nonpivot cols.) = total # cols. in A

rank(A) + nullity(A) = n

⑥ Systems (of Linear Eqs.)

Homogeneous: $A\vec{x} = \vec{0}$

Sol'n set is a subspace, namely $N(A)$.

Nonhomogeneous: $A\vec{x} = \vec{b}$ $(\vec{b} \neq \vec{0})$

What's an easy way of seeing

Sol'n set is not a subspace.
$\vec{0}$ is not in it, since $A\vec{0} = \vec{0}$

Same A as before

Ex Solve $A\vec{x} = \vec{b}$ where

$$\left[\begin{array}{cccc|c} 1 & 0 & 3 & -1 & 2 \\ 0 & 2 & 4 & 0 & 4 \\ 2 & 1 & 8 & -2 & 6 \end{array}\right]$$

same as before

will change

$\downarrow$ Gauss-Jordan

$$\left[\begin{array}{cccc|c} 1 & 0 & 3 & -1 & 2 \\ 0 & 1 & 2 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Same RRE form as before

not a pivot col.
$\Rightarrow$ consistent

doesn't have a leading non-0 entry

Sol'n Set in Parametric Form

$$\begin{cases} x_1 = 2 - 3t + u \\ x_2 = 2 - 2t \\ x_3 = \quad\ t \\ x_4 = \qquad u \\ t, u \text{ are any real \#s} \end{cases}$$

same as for $A\vec{x} = \vec{0}$

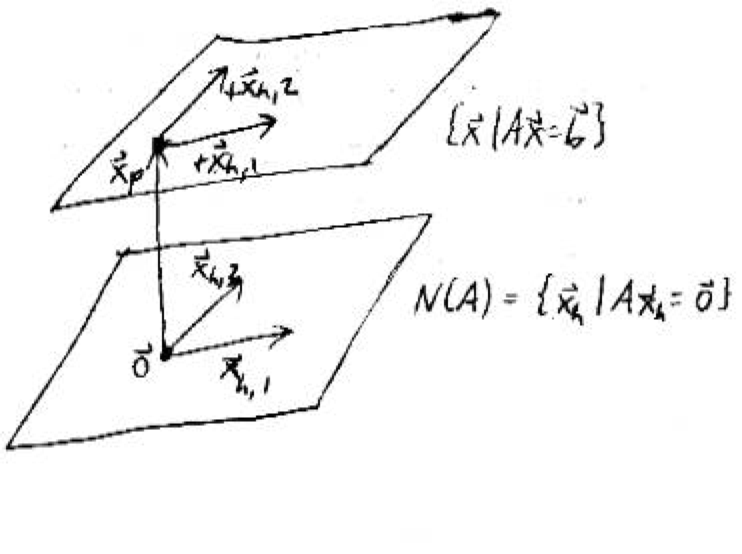
→ Vector Notation

pushes $N(A)$ away from $\vec{0}$ (origin)

constant terms

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \underbrace{\begin{bmatrix} 2 \\ 2 \\ 0 \\ 0 \end{bmatrix}}_{\substack{\vec{x}_p, \\ \text{a particular} \\ \text{sol'n of} \\ A\vec{x} = \vec{b} \\ (t=0,\, u=0)}} + \underbrace{t \begin{bmatrix} -3 \\ -2 \\ 1 \\ 0 \end{bmatrix} + u \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}}_{\substack{\text{sweep out } N(A) \\ (t, u \text{ are any real \#s}) \\ = \{\vec{x}_h \mid A\vec{x}_h = \vec{0}\}}}$$

If consistent,
Sol'n set for $A\vec{x} = \vec{b}$ is "$\parallel$" to $\underbrace{N(A)}_{\text{sol'n set for } A\vec{x} = \vec{0}}$.



(H) $A\vec{x}=\vec{b}$ is consistent $\leftrightarrow$ $\vec{b}$ is in $Col(A)$

Why?

$$A\vec{x}=\vec{b} \text{ consistent}$$

$$\leftrightarrow \begin{bmatrix}\vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n\end{bmatrix}\underbrace{\begin{bmatrix}x_1\\x_2\\\vdots\\x_n\end{bmatrix}}_{\text{weights}}=\vec{b}$$

$$\overset{\text{Trust me}}{\Longleftrightarrow} \underbrace{x_1\vec{a}_1+x_2\vec{a}_2+\dots+x_n\vec{a}_n}_{\text{a linear combo of the cols. of } A}=\vec{b}$$

$$\leftrightarrow \vec{b} \text{ is in } \underbrace{Span(\{\text{cols. of } A\})}_{Col(A)}$$

(old) Ex

$$\left[\begin{array}{cccc|c} 1 & 0 & 3 & -1 & 2\\ 0 & 2 & 4 & 0 & 4\\ 2 & 1 & 8 & -2 & 6\\ \end{array}\right]$$

A = ($\vec{a}_1$ $\vec{a}_2$ $\vec{a}_3$ $\vec{a}_4$), $\vec{b}$

One sol'n was $\vec{x}_p=\begin{bmatrix}2\\2\\0\\0\end{bmatrix}$

We're going to weight the cols of A by the coords of the sol'n.

$$(x_1\vec{a}_1+x_2\vec{a}_2+x_3\vec{a}_3+x_4\vec{a}_4)$$

$$=2\begin{bmatrix}1\\0\\2\end{bmatrix}+2\begin{bmatrix}0\\2\\1\end{bmatrix}+\cancel{0[\ ]}+\cancel{0[\ ]}$$

$$=\begin{bmatrix}2\\4\\6\end{bmatrix}\leftarrow\vec{b}\ !!$$