

CH. 5: INNER PRODUCT SPACES (\mathbb{R}^n)S.I.: LENGTH and DOT PRODUCT in \mathbb{R}^n ① $\|\vec{v}\|$ If $\vec{v} = (v_1, v_2, \dots, v_n)$, then $\|\vec{v}\| = \text{the length or magnitude or norm of } \vec{v}$

$$= \sqrt{v_1^2 + v_2^2 + \dots + v_n^2}$$

(Euclidean norm,
or ℓ^2 -norm) p -norm: $(\sum |v_i|^p)^{\frac{1}{p}}$

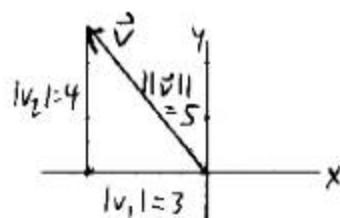
l-norm: grid norm
 unit sphere: octahedron
 ∞ -norm: max norm, l-infinity norm

sphere: cube
 $\|\vec{v}\|_1 \geq \|\vec{v}\|_2 \geq \|\vec{v}\|_\infty$

Meyer 274.5

Ex If $\vec{v} = (-3, 4)$

$$\|\vec{v}\| = \sqrt{(-3)^2 + (4)^2} = 5$$



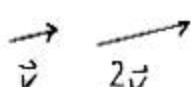
Pyth. Thm.

Props.

① $\|\vec{v}\| \geq 0$

② $\|\vec{v}\| = 0 \leftrightarrow \vec{v} = \vec{0}$

③ $\|cv\| = |c| \|\vec{v}\|$
 scalar stretching factor



\vec{u} is a unit vector $\Leftrightarrow \|\vec{u}\| = 1$

The unit vector in the direction of \vec{v} ($\vec{v} \neq \vec{0}$) is

$$\vec{u} = \frac{1}{\|\vec{v}\|} \vec{v} \text{ or } \frac{\vec{v}}{\|\vec{v}\|}$$

"normalizing \vec{v} "

Ex Normalize $\vec{v} = (1, 0, -4)$

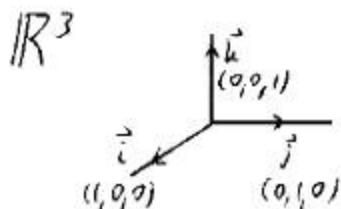
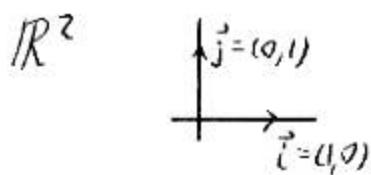
$$\vec{u} = \frac{1}{\|\vec{v}\|} \vec{v} = \frac{1}{\sqrt{(1)^2 + (0)^2 + (-4)^2}} (1, 0, -4)$$

$$= \frac{1}{\sqrt{17}} (1, 0, -4)$$

$$= \left(\frac{1}{\sqrt{17}}, 0, -\frac{4}{\sqrt{17}} \right)$$

Up to 2!
Don't have to
normalize vs.

Standard unit vectors in \mathbb{R}^n
are in the standard basis

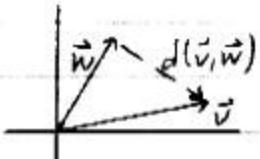


book uses
 \vec{v}, \vec{w}
high...

(B) $d(\vec{v}, \vec{w})$

$$\begin{aligned} d(\vec{v}, \vec{w}) &= \text{the distance between } \vec{v} \text{ and } \vec{w} \\ &= \|\vec{v} - \vec{w}\| \\ &= \sqrt{(v_1 - w_1)^2 + \dots + (v_n - w_n)^2} \end{aligned}$$

distance
bet. the 2
terminal pts.



Ex If $\vec{v} = (-1, 0, 2)$ and $\vec{w} = (-3, 2, -1)$,

$$\begin{aligned} d(\vec{v}, \vec{w}) &= \|\vec{v} - \vec{w}\| \\ &= \|(2, -2, 3)\| \\ &= \sqrt{(2)^2 + (-2)^2 + (3)^2} \\ &= \boxed{\sqrt{17}} \end{aligned}$$

Props

- ① $d(\vec{v}, \vec{w}) \geq 0$
- ② $d(\vec{v}, \vec{w}) = 0 \Leftrightarrow \vec{v} = \vec{w}$
- ③ $d(\vec{v}, \vec{w}) = d(\vec{w}, \vec{v})$

is defined as
a length...

up to 25

① $\vec{v} \cdot \vec{w}$

$\vec{v} \cdot \vec{w}$ = the dot product of \vec{v} and \vec{w}
 (Euclidean inner product for \mathbb{R}^n)

$$= v_1 w_1 + v_2 w_2 + \dots + v_n w_n \quad \langle \vec{v}, \vec{w} \rangle$$

is a scalar

Ex If $\vec{v} = (0, 1, 2)$ and $\vec{w} = (-3, 4, 5)$, then

$$\begin{aligned}\vec{v} \cdot \vec{w} &= (0)(-3) + (1)(4) + (2)(5) \\ &= \boxed{14}\end{aligned}$$

There are
other IPs, and
you can have
IPs on other
spaces.

Outer product
of \mathbb{C} and \mathbb{R}
 $[\]c \ [] = [\]$
 Me: "multi-table"
 Meyer 10.3

Inner product \rightarrow
 Standard IP for
 matrices:
 $\langle A, B \rangle = \text{trace}(A^T B)$
 Meyer 2.86

In \mathbb{C} : $(x, y) = x^* y$
 $= \sum_i x_i y_i$
 complex conj.

Conjugate 1 vec
 1st, then 2nd
 up to 29

Props

- ① $\vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v}$ ("." is comm.)
- ② $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$ ("." distributes over "+")
- ③ $(c\vec{v}) \cdot \vec{w} = c(\vec{v} \cdot \vec{w})$ (You can "pull out" scalars)
- ④ $\vec{v} \cdot \vec{v} = \|\vec{v}\|^2$ (so ≥ 0)

Ex Evaluate $\vec{v} \cdot (4\vec{v} + \vec{w})$ if $\|\vec{v}\| = 3$, $\vec{v} \cdot \vec{w} = 2$

$$\begin{aligned}&= (\vec{v} \cdot 4\vec{v}) + (\vec{v} \cdot \vec{w}) \\&= 4(\vec{v} \cdot \vec{v}) + (\vec{v} \cdot \vec{w}) \\&= 4\|\vec{v}\|^2 + 2 \\&= 4(3)^2 + 2 \\&= \boxed{38}\end{aligned}$$

Up to 33

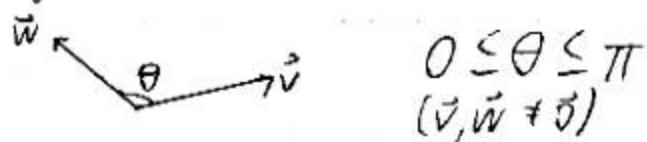
More: Exs. 5, 6 (pp. 256-7)

Note If \vec{v}, \vec{w} are column vectors in \mathbb{R}^n ,

$$\vec{v} \cdot \vec{w} = \underbrace{\vec{v}^T}_{= \vec{v}} \underbrace{\vec{w}}_{= \vec{w}}$$

How can I rep
 using Ch. 2 not?

① Angles

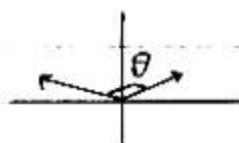


$$\cos \theta = \frac{\vec{v} \cdot \vec{w}}{\|\vec{v}\| \|\vec{w}\|}$$

Is between $-1, +1$ by
Cauchy-Schwarz Inequality

$$|\vec{v} \cdot \vec{w}| \leq \|\vec{v}\| \|\vec{w}\|$$

Ex $\vec{v} = (2, 1), \vec{w} = (-3, 1)$



$$\cos \theta = \frac{(2, 1) \cdot (-3, 1)}{\|(2, 1)\| \ \|(-3, 1)\|}$$

$$= \frac{2(-3) + 1(1)}{\sqrt{(2)^2 + (1)^2} \ \sqrt{(-3)^2 + (1)^2}}$$

$$= \frac{-5}{\sqrt{5} \sqrt{10}}$$

$$(\approx -0.7071)$$

May relate to norm.

(Cauchy: most general in terms of naming
 2nd most prolific (Gauss)
 or C-Bunyakowski-S
 cont. funcns.
 Meyer)

Ineqn does not exceed den.

technically,
should be in
rads
up to 47

$$\theta \approx \cos^{-1}(0.7071) \text{ (RAD)}$$

$$\theta = 135^\circ \quad \text{RAD vs. DEG}$$

E $\vec{v} \perp \vec{w}$

$\vec{v} \perp \vec{w}$ means " \vec{v} is orthogonal to \vec{w} "

$$\Leftrightarrow \vec{v} \cdot \vec{w} = 0$$

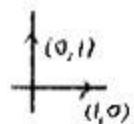
Why?  $\vec{v}, \vec{w} \neq 0$

$$\underbrace{\cos(90^\circ)}_0 = \frac{\vec{v} \cdot \vec{w}}{\|\vec{v}\| \|\vec{w}\|}$$

$$\Leftrightarrow \vec{v} \cdot \vec{w} = 0$$

$\vec{0} \perp$ every vector in \mathbb{R}^n .

Ex



$$(1,0) \cdot (0,1) = 0$$

Ex Determine all vectors in \mathbb{R}^3 that are $\perp \vec{v} = (1, 2, 0)$.

$$\begin{aligned} \vec{v} \cdot \vec{w} &= (1, 2, 0) \cdot \underbrace{(w_1, w_2, w_3)}_{\text{Generic form}} \\ &= w_1 + 2w_2 \stackrel{!}{=} 0 \end{aligned}$$

$$\begin{aligned} w_1 &= -2w_2 \\ w_2 &\text{ is free ("t")} \\ w_3 &\text{ is free ("u")} \end{aligned}$$

(alc 3: given a plane, find a normal.
Here, given a normal, determine a plane thru it.
in param. form.
 $x+2y=0$
picture?!

$$w_1 = -2t$$

$$w_2 = t$$

$$w_3 = u$$

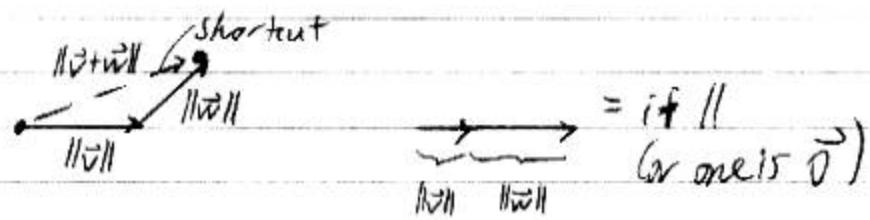
t, u are any real #s

up to 64



⑤ Triangle Inequality

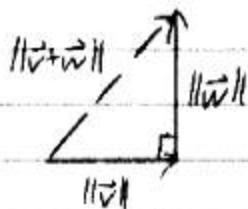
$$\|\vec{v} + \vec{w}\| \leq \|\vec{v}\| + \|\vec{w}\|$$



up to 66

⑥ Pythagorean Theorem

$$\vec{v} \perp \vec{w} \Leftrightarrow \|\vec{v} + \vec{w}\|^2 = \|\vec{v}\|^2 + \|\vec{w}\|^2$$



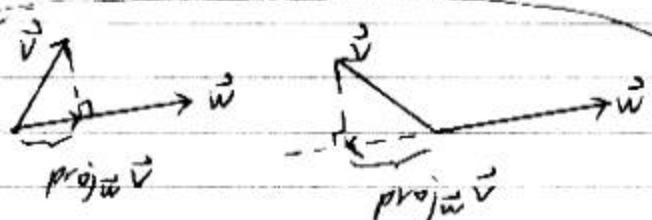
Actually, Inner
Product spaces 5.2: (ORTHOGONAL PROJECTIONS in \mathbb{R}^n)

pp. 270-272

$\text{proj}_{\vec{w}} \vec{v}$ = the orthogonal projection of
 \vec{v} onto \vec{w} ($\vec{w} \neq \vec{0}$)

$$= \underbrace{\left(\frac{\vec{v} \cdot \vec{w}}{\vec{w} \cdot \vec{w}} \right) \vec{w}}_{\#} \quad (\leftarrow \text{I will give this to you on test})$$

is the scalar mult. : of \vec{w}
"closest" to \vec{v}

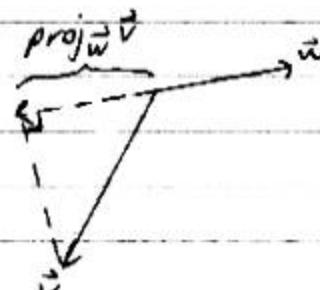


Ex Find the orthog. proj. of
 $\vec{v} = (-2, -2, -3)$ onto $\vec{w} = (1, 3, 2)$

$$\text{proj}_{\vec{w}} \vec{v} = \left[\frac{(-2, -2, -3) \cdot (1, 3, 2)}{(1, 3, 2) \cdot (1, 3, 2)} \right] (1, 3, 2)$$

$$= \underbrace{\frac{-14}{14}}_{-1} (1, 3, 2)$$

$$= (-1, -3, -2)$$



5.3: ORTHONORMAL BASES in \mathbb{R}^n , GRAM-SCHMIDT PROCESS

(A) Definitions

A set of vectors is orthogonal $\xrightarrow{\text{OG}}$ every pair is.

$$\textcircled{1} \quad \vec{v}_i \cdot \vec{v}_j = 0 \quad \text{for all } i, j \quad (\text{if } \vec{v}_i \text{ and } \vec{v}_j \text{ are orthogonal})$$

pairwise OG

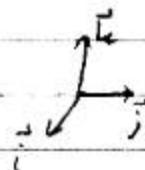
also orthonormal (ON)

if each vector is a unit vector.

\textcircled{1} and

$$\textcircled{2} \quad \|\vec{v}_i\| = 1 \quad \text{for all } i$$

Ex Standard basis in \mathbb{R}^3 is ON.



(B) Any orthogonal (OG) set of nonzero vectors is LI.

Proof Assume $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\}$ is OG; each " $\vec{v}_i \neq \vec{0}$ "

Show $c_1 \vec{v}_1 + c_2 \vec{v}_2 + \dots + c_k \vec{v}_k = \vec{0}$

is only solved by $c_1 = c_2 = \dots = c_k = 0$

$$c_1 \vec{v}_1 + c_2 \vec{v}_2 + \dots + c_k \vec{v}_k = \vec{0}$$

$$\Rightarrow \vec{v}_1 \cdot (c_1 \vec{v}_1 + c_2 \vec{v}_2 + \dots + c_k \vec{v}_k) = \vec{v}_1 \cdot \vec{0}$$

$$\Rightarrow c_1 (\vec{v}_1 \cdot \vec{v}_1) + c_2 (\vec{v}_1 \cdot \vec{v}_2) + \dots + c_k (\vec{v}_1 \cdot \vec{v}_k) = 0$$

$$\stackrel{\|\vec{v}_1\|^2}{=} 0 \text{ by } \perp \quad \stackrel{=0}{\Rightarrow} \text{ by } \perp$$

$$\Rightarrow c_1 \underbrace{\|\vec{v}_1\|^2}_{\neq 0 \text{ Lec. } \vec{v}_1 \neq \vec{0} \text{ by assumption}} = 0$$

Concl.

By successively
by \vec{v}_2, \dots

$\Rightarrow c_1 = 0$

Repeat using $\vec{v}_2, \dots, \vec{v}_k$
 $c_2 = 0 \quad \dots \quad c_k = 0$

① Any OG set of n nonzero vectors in \mathbb{R}^n
is a basis for \mathbb{R}^n .

Why? Such a set is LI. ②

for a set of n vectors in \mathbb{R}^n :

Span \mathbb{R}^n ↗ if you have one,
LI ↙ you have both
Basis ✓

Read Ex 4 (p.279)

You'd need a
hint.
3-timer, 6F
maybe faster.

Verify $S = \{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ is a basis for \mathbb{R}^4
in \mathbb{R}^4

- ✓ Each $\vec{v}_i \neq \vec{0}$
- ✓ $(\vec{v}_i \cdot \vec{v}_j = 0 \text{ for all } i \neq j)$
pairwise OG

So, S is an OG basis for \mathbb{R}^4 .

① Why are orthonormal (ON) bases so cool?

lectures 5:

If $B = \{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_m\}$ is an ON basis
for \mathbb{R}^n , then for any \vec{w} in \mathbb{R}^n ,

c_i : Fourier
coeffs
Fourier dealt w/
OG-frm. IP space
w/ trig. func. f has
oblique on 1st level
of Little Fourier
Mayer 299 ff

$$\vec{w} = \underbrace{c_1 \vec{u}_1 + c_2 \vec{u}_2 + \dots + c_n \vec{u}_n}_{\text{orthog. proj.}} \quad (\text{Fourier expansion})$$

$$c_1 = \frac{\langle \vec{w}, \vec{u}_1 \rangle}{\langle \vec{u}_1, \vec{u}_1 \rangle} \vec{u}_1$$

$$c_{i+1} \{ \vec{u}_2 \vec{u}_3 \vec{u}_4 \} \vec{u}_i$$

For any OG basis, we can
decompose \vec{w} into a sum
of orthog. projections onto
the basis vectors.

where $c_i = \vec{w} \cdot \vec{u}_i \quad i=1, 2, \dots, n$

Take \vec{w} of original vector wrt each of the basis vectors.

coordinates of \vec{w}

with respect to B

We don't have to solve a system!

Can rationalize
the denominator.

$$\text{Ex } B = \left\{ \underbrace{\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right)}_{\vec{u}_1}, \underbrace{\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right)}_{\vec{u}_2} \right\}$$

is an ON basis for \mathbb{R}^2 . 

Find the coords of $\vec{w} = (1, 2)$ (with respect to) B , " $[\vec{w}]_B$ " wrt

Ch. 4 Way

$$\text{Solve } \left[\begin{array}{cc|c} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 1 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 2 \\ \hline \vec{u}_1 & \vec{u}_2 & \vec{w} \end{array} \right]$$

Now, easier!

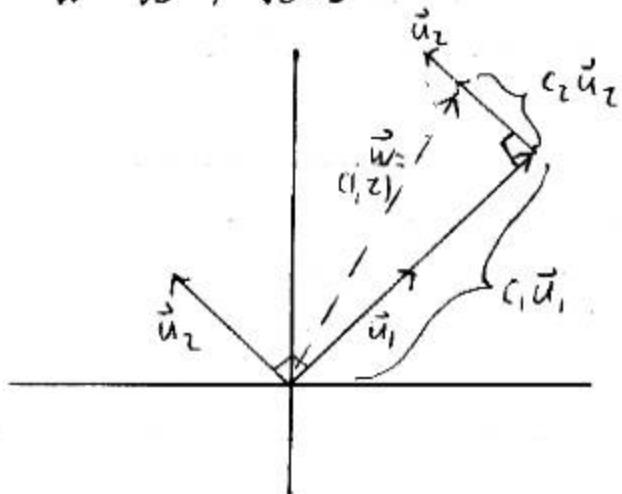
$$\begin{aligned} c_1 &= \vec{w} \cdot \vec{u}_1 \\ &= (1, 2) \cdot \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right) \\ &= \frac{3}{\sqrt{2}} \quad (\approx 2.1) \end{aligned}$$

$$\begin{aligned} c_2 &= \vec{w} \cdot \vec{u}_2 \\ &= (1, 2) \cdot \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right) \\ &= \frac{1}{\sqrt{2}} \quad (\approx 0.7) \end{aligned}$$

5.3.4

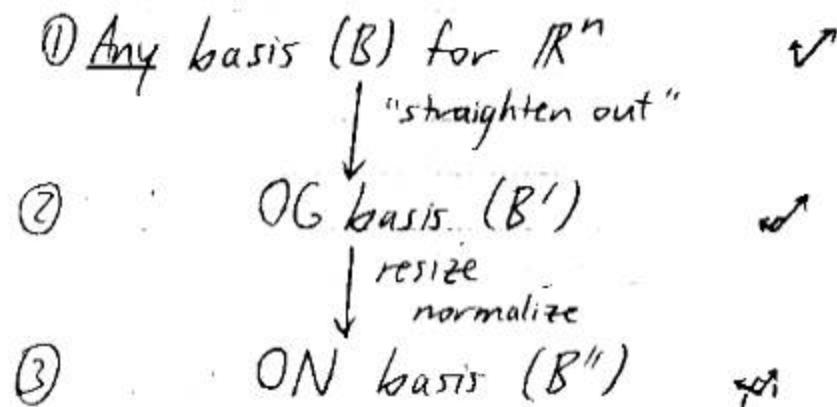
$$[\vec{w}]_B = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} \frac{3}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \approx \begin{bmatrix} 2.1 \\ 0.7 \end{bmatrix}$$

$$\vec{w} = \frac{3}{\sqrt{2}} \vec{u}_1 + \frac{1}{\sqrt{2}} \vec{u}_2$$



E) Gram-Schmidt Orthonormalization

Idea



Process

① Let $B = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ be any basis for \mathbb{R}^n .

② Let $B' = \{\vec{w}_1, \vec{w}_2, \dots, \vec{w}_n\}$ where

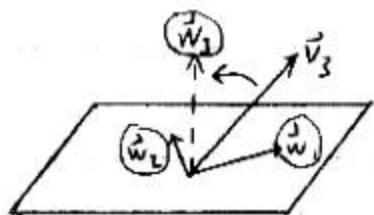
$$\vec{w}_1 = \vec{v}_1 \quad \rightarrow \text{Grab } \vec{v}_1$$

$$\vec{w}_2 = \vec{v}_2 - \underbrace{\text{proj}_{\vec{w}_1} \vec{v}_2}_{\left(\frac{\vec{v}_2 \cdot \vec{w}_1}{\vec{w}_1 \cdot \vec{w}_1} \right) \vec{w}_1} \quad (\leftarrow \text{I'll give you this formula on a test.})$$



But not this
not that generous

$$\vec{w}_3 = \vec{v}_3 - \underbrace{\text{proj}_{\vec{w}_1} \vec{v}_3}_{\left(\frac{\vec{v}_3 \cdot \vec{w}_1}{\vec{w}_1 \cdot \vec{w}_1} \right) \vec{w}_1} - \underbrace{\text{proj}_{\vec{w}_2} \vec{v}_3}_{\left(\frac{\vec{v}_3 \cdot \vec{w}_2}{\vec{w}_2 \cdot \vec{w}_2} \right) \vec{w}_2}$$

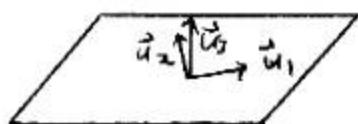


etc.

③ Let B'' = the ON basis
 $\{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_n\}$ where

$$\vec{u}_i = \frac{\vec{w}_i}{\|\vec{w}_i\|} \quad i = 1, \dots, n$$

$$\left(\vec{u}_n = \frac{\vec{w}_n}{\|\vec{w}_n\|} \right)$$



Ex Use G-S to transform the \mathbb{R}^3 -basis
 $B = \left\{ \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 3 \end{bmatrix} \right\}$ into an
 ON basis.

Find $B' = \{\vec{w}_1, \vec{w}_2, \vec{w}_3\}$ ⑥

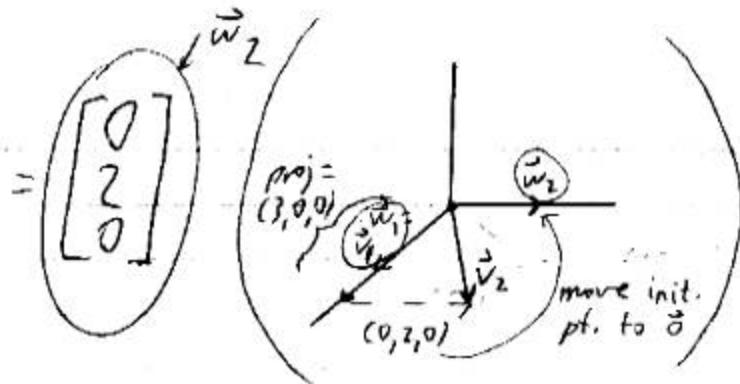
$$\vec{w}_1 = \vec{v}_1 = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$$

$$\vec{w}_2 = \vec{v}_2 - \underbrace{\text{proj}_{\vec{w}_1} \vec{v}_2}_{\left(\frac{\vec{v}_2 \cdot \vec{w}_1}{\vec{w}_1 \cdot \vec{w}_1} \right) \vec{w}_1}$$

$$= \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} - \frac{(3, 2, 0) \cdot (2, 0, 0)}{(2, 0, 0) \cdot (2, 0, 0)} \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$$

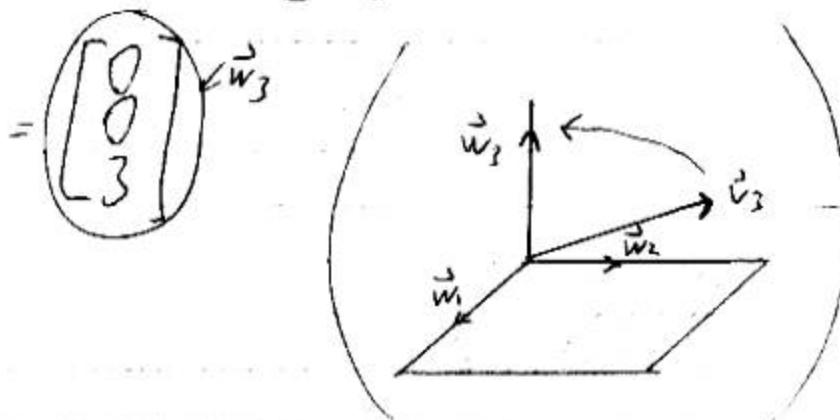
$$= \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} - \frac{6}{4} \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} + \begin{bmatrix} -3 \\ 0 \\ 0 \end{bmatrix}$$



$$\vec{w}_3 = \vec{v}_3 - \underbrace{\text{proj}_{\vec{w}_1} \vec{v}_3}_{\frac{\vec{v}_3 \cdot \vec{w}_1}{\vec{w}_1 \cdot \vec{w}_1} \vec{w}_1} - \underbrace{\text{proj}_{\vec{w}_2} \vec{v}_3}_{\frac{\vec{v}_3 \cdot \vec{w}_2}{\vec{w}_2 \cdot \vec{w}_2} \vec{w}_2}$$

$$= \begin{bmatrix} 3 \\ 4 \\ 3 \end{bmatrix} - \frac{6}{4} \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} - \frac{8}{4} \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$$



Find $B'' = \{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ ON

(Can tell)

$$\vec{u}_1 = \frac{\vec{w}_1}{\|\vec{w}_1\|} = \frac{(2, 0, 0)}{\sqrt{(2)^2 + (0)^2 + (0)^2}} = \frac{(2, 0, 0)}{2} = \boxed{(1, 0, 0)}$$

$$\vec{u}_2 = \frac{\vec{w}_2}{\|\vec{w}_2\|} = \boxed{(0, 1, 0)}$$

$$\vec{u}_3 = \frac{\vec{w}_3}{\|\vec{w}_3\|} = \boxed{(0, 0, 1)}$$

Usually, you'll
get a rotation
of the sb.
(if start with free (unfixed)
→ diff ON basis)

Of course, the standard
basis for \mathbb{R}^3 works.

So what's the point?

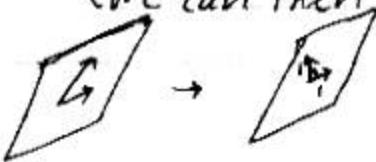
Key Applic.

Given a basis for a subspace of \mathbb{R}^n

↓ G-S

Get an ON basis for that subspace

(We can then compute coordinates easily.)



relative to that
ON basis

Meyer 314-7
Meyer 36-5 not good numerically.
Modified version better.
numerically

Meyer 311
QR "records"
G-S \approx
LU "records"
GE
A can be skinny!

G-S \rightarrow QR fac'n of A w/LI cols.

ON cols. upper tri.

"records" G-S, as
LU fac'n "records" Gaussian elimination