Puts some dynamism in L.A. (not that it wasn't there before...)

CH. 6: LINEAR TRANSFORMATIONS

6.1: INTRO

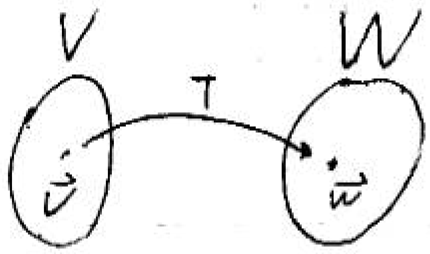
(A) Terminology

W doesn't have to be subspace

Let V, W be VSs

$T : V \longrightarrow W$ means:

The function T that maps the <u>domain</u> V into the <u>codomain</u> W.



$T(\vec{v}) = \vec{w}$

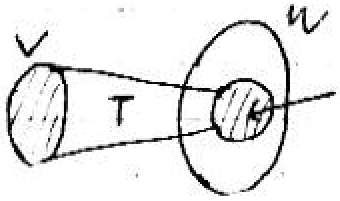
$\vec{v}$ in the <u>preimage</u> of $\vec{w}$

$\vec{w}$ the <u>image</u> of $\vec{v}$

maybe $\rightrightarrows \vec{w}$

no one $\to \vec{w}$

<u>Range</u> of $T = \{T(\vec{v}) \mid \vec{v} \text{ is in } V\}$



Range(T) = set of all images

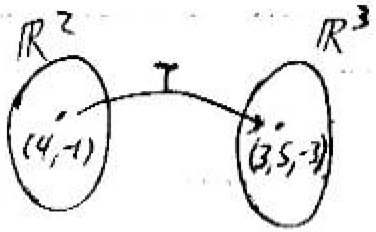
(B) Ex

For any $\vec{v} = (v_1, v_2)$ in $\mathbb{R}^2$, let $T: \mathbb{R}^2 \to \mathbb{R}^3$ be defined by $T(v_1, v_2) = (v_1 + v_2, v_1 - v_2, 3v_2)$

a) Find the image of $\vec{v} = (4, -1)$

$$\begin{aligned} T(\vec{v}) &= T(4, -1) \\ &= (4 + (-1), 4 - (-1), 3(-1)) \\ &= (3, 5, -3) \end{aligned}$$

Only $(4,-1) \to \vec{w}$
$3v_2 = -3 \to v_2 = -1$
$v_1 + v_2 = 3 \to v_1 = 4$



b) Find the preimage of $\vec{w} = (4, 2, 3)$

Find all $\vec{v}$ such that

$$T(\vec{v}) = \vec{w}$$
$$(v_1 + v_2, v_1 - v_2, 3v_2) = (4, 2, 3)$$

Here, system easy.

$$\begin{cases} v_1 + v_2 = 4 \\ v_1 - v_2 = 2 \\ \quad\;\; 3v_2 = 3 \end{cases}$$

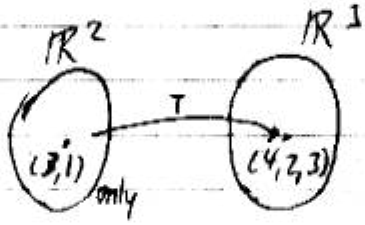
$$\left[\begin{array}{cc|c} 1 & 1 & 4 \\ 1 & -1 & 2 \\ 0 & 3 & 3 \end{array}\right]$$
$$\underbrace{\qquad\qquad}_{A}$$

We get the unique sol'n

$v_1 = 3$

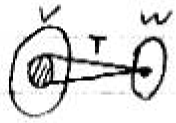
$v_2 = 1$

$\{(3,1)\}$



up to 5

If ∞ many sol'ns → parametrize sol'n set



Ⓒ Linear Transformations (LTs)

$T: V \to W$ is a LT if, for all $\vec{v}_1, \vec{v}_2$ in V and any scalar c,

Image of sum = sum of images

Can add, then map or map, then add → same result

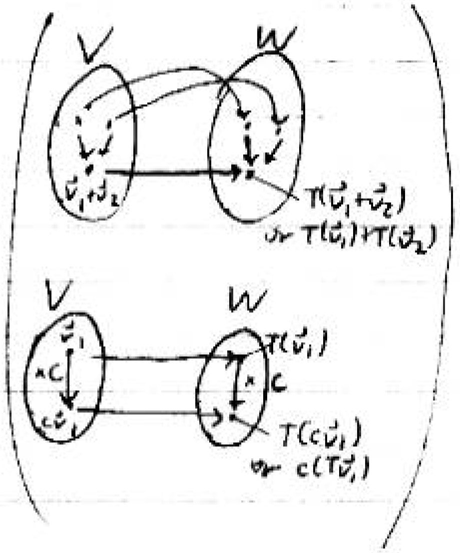
① $T(\vec{v}_1 + \vec{v}_2) = \underbrace{T(\vec{v}_1) + T(\vec{v}_2)}_{\text{map to } W\text{, 1st then, add in } W}$

add in V, 1st then, map to W

You can rescale before or after T is applied → same result.

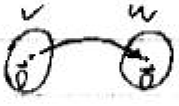
② $T(c\vec{v}_1) = c\,\underbrace{T(\vec{v}_1)}_{\text{map to } W\text{, 1st then, } \times c}$

×c in V, 1st then, map to W



Key Props.

$T(\vec{0}_V) = \vec{0}_W$



✪ $T(c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_n\vec{v}_n) = c_1 T(\vec{v}_1) + \dots + c_n T(\vec{v}_n)$

"$\vec{v}_i$"s are in V

This leads to...

① LTs and Bases

If we know $T(\vec{b}_1), T(\vec{b}_2), \ldots, T(\vec{b}_n)$ for any one basis $\{\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n\}$ of V, then we can find $T(\vec{v})$ for any $\vec{v}$ in V.

If we know how T maps a basis of V, we know how T maps everything in V.

Ex Let $T: \mathbb{R}^2 \to \mathbb{R}^3$ be a LT such that

$$T(1,0) = (1,3,-2)$$
$$T(0,1) = (4,2,0)$$

Find $T(-3,2)$

You don't have to do this

I'll be switching between row vecs, col vecs - it often doesn't matter in this material here.

$$T\left(\begin{bmatrix}-3\\2\end{bmatrix}\right) = T\left(\begin{bmatrix}-3\\0\end{bmatrix} + \begin{bmatrix}0\\2\end{bmatrix}\right)$$

$$= T\left(-3\begin{bmatrix}1\\0\end{bmatrix} + 2\begin{bmatrix}0\\1\end{bmatrix}\right)$$

$$\overset{☆}{=} -3T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) + 2T\left(\begin{bmatrix}0\\1\end{bmatrix}\right)$$

$$= -3(1,3,-2) + 2(4,2,0)$$

$$= (-3,-9,6) + (8,4,0)$$

$$= (5,-5,6)$$

You can find any $T(v_1, v_2)$

(E) The L.T Given by a Matrix

$$T(\vec{v}) = \overset{m\times n}{A}\vec{v} \text{ defines a L.T}$$

$$T: \mathbb{R}^n \to \mathbb{R}^m$$

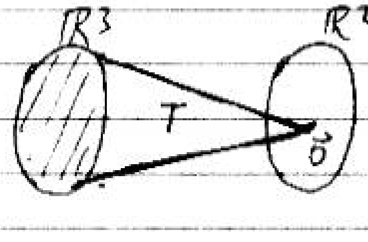
$$\begin{array}{cc} & n \\ m & \begin{bmatrix} & A & \end{bmatrix} \end{array} \underbrace{\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}}_{\vec{v} \text{ in } \mathbb{R}^n} = \underbrace{\begin{bmatrix} w_1 \\ \vdots \\ w_m \end{bmatrix}}_{T(\vec{v}) \text{ in } \mathbb{R}^m}$$

(column vectors)

<u>Ex</u> If $A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{2\times 3}$, then

$T(\vec{v}) = A\vec{v}$ defines the <u>zero transformation</u>

$$T: \mathbb{R}^3 \to \mathbb{R}^2$$

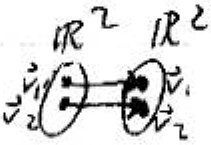


Ex If $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, then

$T(\vec{v}) = A\vec{v}$ defines the identity transformation

$T: \mathbb{R}^2 \to \mathbb{R}^2$

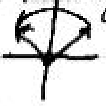
Here, $T(\vec{v}) = \vec{v}$ for all $\vec{v}$ in $V = W = \mathbb{R}^2$



(Do Last)

Ex If $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, then (see Section 12.4, Calc II, Swokowski on rotated conics)

$T: \mathbb{R}^2 \to \mathbb{R}^2$ rotates every vector in $\mathbb{R}^2$ counterclockwise about the origin.

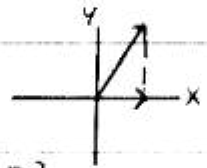


(Optional proof - p.332)

Ex If $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, then

$T: \mathbb{R}^2 \to \mathbb{R}^2$ is a projection in $\mathbb{R}^2$.

$T(\vec{v}) = A\vec{v} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ 0 \end{bmatrix}$



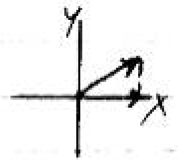
Ex 8 - p.332: $\mathbb{R}^3$

<u>Ex</u> Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

let $T(\vec{v}) = A\vec{v}$ define $T: \mathbb{R}^2 \to \mathbb{R}^2$

a) Find the image of $(4,3)$.

$$T(4,3) = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 4 \\ 3 \end{bmatrix}}_{\vec{v}} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$$



b) Find the premage of $(3,0)$

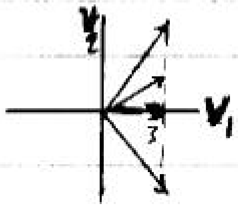
$$T(\vec{v}) = A\vec{v}$$
$$A\vec{v} = T(\vec{v})$$

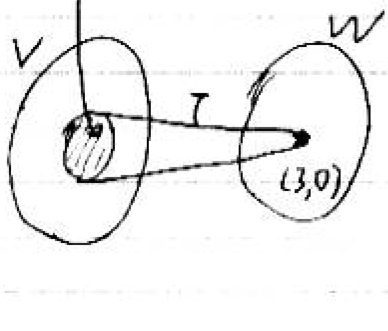
$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

$$\left(\left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 0 & 0 \end{array}\right]\right)$$

$v_1 = 3$
$v_2 = t$ (free)

$\{(3,t) \mid t \text{ is a real } \#\}$



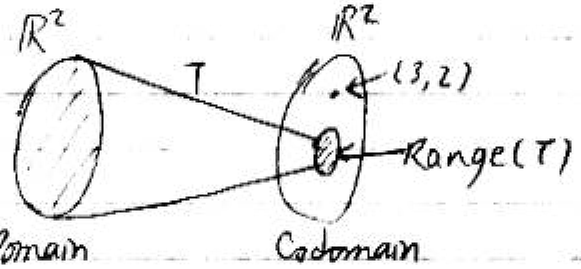


c) Find the *preimage* of (3,2)

$$\left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 0 & 2 \end{array}\right]$$

Preimage = ϕ

He doesn't get to play. He's left out.



Go back to rotation Ex (6.1.6)

6.2: KERNEL + RANGE

Let $T: V \to W$ be a L.T.

(A) Ker (T)

$=$ kernel of T

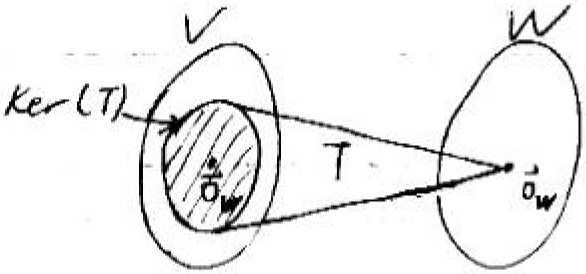
$= \{\vec{v} \text{ in } V \mid T(\vec{v}) = \vec{0}_w\}$

$=$ set of vectors in V that T maps to $\vec{0}$ in W.

is a subspace of V (pf. pp. 339-40)

what vector in V must be in the kernel?

subspace



Ex $T: \mathbb{R}^2 \to \mathbb{R}^2$, find Ker (T) if

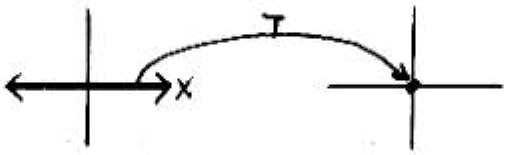
$T(x,y) = \underbrace{(0,y)}_{\text{what makes this} = \vec{0}?}$ ← projection on y-axis

$$\begin{cases} \cancel{0=0} \\ y=0 \end{cases}$$

Your book sticks w/x

$$\begin{cases} x = t \text{ (free)} \\ y = 0 \end{cases}$$

$$\mathrm{Ker}(T) = \{(t,0) \mid t \text{ is a real } \#\}$$
$$= x\text{-axis}$$



$\underline{T(\vec{x}) = \overset{m\times n}{A}\vec{x} \text{ defines } T:\mathbb{R}^n \to \mathbb{R}^m}$ (A LT can be given by a matrix.)

$$\mathrm{Ker}(T) = \{\vec{x} \text{ in } \mathbb{R}^n \mid A\vec{x} = \vec{0}\}$$
$$= N(A), \text{ the } \underline{\text{nullspace}} \text{ of } A$$

Its dimension $=$ nullity(T)
or nullity(A)

Ex $T: \mathbb{R}^3 \to \mathbb{R}^2$
$T(\vec{x}) = A\vec{x}$, $\vec{x}$ in $\mathbb{R}^3$

$$A = \begin{bmatrix} 2 & 4 & 6 \\ 0 & 1 & 3 \end{bmatrix}$$

Find a basis for Ker(T) as a subspace of $\mathbb{R}^3$.

Solution

Solve $A\vec{x} = \vec{0}$ and find a basis for $N(A)$.

$$\left[\begin{array}{ccc|c} 2 & 4 & 6 & 0 \\ 0 & 1 & 3 & 0 \end{array}\right]$$

$$\overset{RRE}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & -3 & 0 \\ 0 & 1 & 3 & 0 \end{array}\right]$$

x_3 free

$$\begin{cases} x_1 \quad\quad - 3x_3 = 0 \\ \quad\quad x_2 + 3x_3 = 0 \end{cases}$$

$$\begin{cases} x_1 = 3x_3 \\ x_2 = -3x_3 \end{cases}$$

$$x_3 \doteq t$$

$$\begin{cases} x_1 = 3t \\ x_2 = -3t \\ x_3 = t \end{cases}$$

$$\vec{x} = t\begin{bmatrix} 3 \\ -3 \\ 1 \end{bmatrix}, \ t \text{ is any real \#} \ \Big\} \text{Ker}(T)\text{: line in } \mathbb{R}^3$$

Basis : $\left\{ \begin{bmatrix} 3 \\ -3 \\ 1 \end{bmatrix} \right\}$

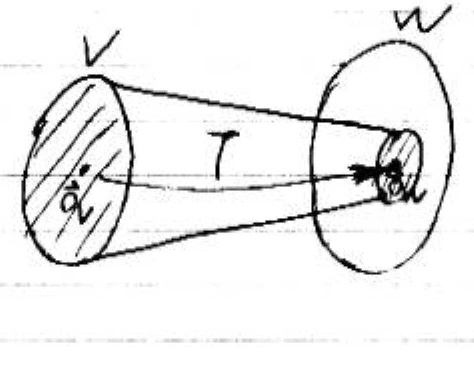
nullity $(T) = 1$

up to 6 Read Exs 1–6

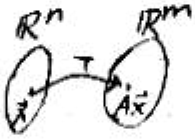
Ⓑ Range(T)

$= \{ T(\vec{v}) \mid \vec{v} \text{ is in } V \}$ (images)

is a subspace of W



$\underline{T(\vec{x}) = \overset{m\times n}{A}\vec{x}}$



Then, $\text{Range}(T) = \{A\vec{x} \mid \vec{x} \text{ is in } \mathbb{R}^n\}$

$= \{\vec{b} \text{ in } \mathbb{R}^m \mid A\vec{x} = \vec{b} \text{ is consistent}\}$

$= \text{Col}(A)$, the <u>column space</u> of A

Its dimension $=$ rank(T) or rank(A)

<u>Ex</u> $T: \mathbb{R}^4 \to \mathbb{R}^3$
$T(\vec{x}) = A\vec{x}$, $\vec{x}$ in $\mathbb{R}^4$

$$A = \begin{bmatrix} 1 & 2 & 4 & 3 \\ 1 & 2 & 5 & 5 \\ 2 & 4 & 8 & 6 \end{bmatrix}$$

Find a basis for Range(T) as a subspace of $\mathbb{R}^3$.

<u>Solution</u>

$\to$ row-echelon "shape"

$$A \sim \begin{bmatrix} 1 & 2 & 4 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Cols 1,3 are pivot cols.

Take cols. 1,3 from A.

$$\text{Basis} = \left\{ \underset{\vec{v}_1}{\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}}, \underset{\vec{v}_2}{\begin{bmatrix} 4 \\ 5 \\ 8 \end{bmatrix}} \right\}$$

Up to 11

Also, $Range(T) = Span(\{\vec{v}_1, \vec{v}_2\}) =$ "plane in $\mathbb{R}^3$"
$rank(T) = 2$

© Sum Formulas

(4.6F) $rank(T) + nullity(T) = n$

$$\dim(Range(T)) + \dim(Ker(T)) = \dim(domain, V)$$

$T(\vec{x}) = A\vec{x}$

$n = \#cols$

$[A] \sim \begin{bmatrix} \text{row-ech} \\ \text{shape} \end{bmatrix}$.

$rank(T) = \#$pivot cols
$nullity(T) = \#$nonpivot cols

Read Exs. 8, 9

from Ker(T)

Ex $T: \mathbb{R}^3 \to \mathbb{R}^2$
$T(\vec{x}) = A\vec{x}$

$$A \sim \begin{bmatrix} 2 & 4 & 6 \\ 0 & 1 & 3 \end{bmatrix}$$

$$\left(\sim \begin{bmatrix} 1 & 8 & -3 \\ 0 & 1 & 3 \end{bmatrix} \right)$$

Be aware of variations of same ?

Up to 35

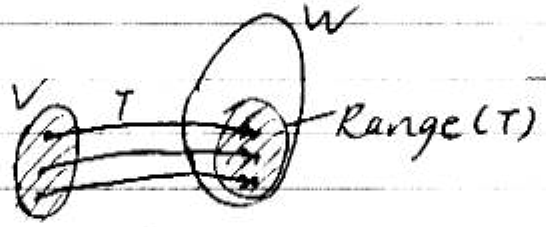
$\text{rank}(T) = \dim(\text{Range}(T)) = 2$

$\text{nullity}(T) = \dim(\text{Ker}(T)) = n - \text{rank}(T)$
$= 3 - 2$
$= 1$

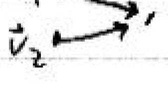
① When is T "One-to-One" (1-1) ?

$\leftrightarrow$ For each $\vec{w}$ in Range (T),

$\leftrightarrow$ whenever $T(\vec{v}_1) = T(\vec{v}_2)$, then $\vec{v}_1 = \vec{v}_2$



NO:



can't have 2 vectors w/ same image

$$\boxed{\text{A LT is 1-1} \Leftrightarrow \mathrm{Ker}(T) = \{\vec{0}_W\}}$$ Proof (optional) p.344

only $\vec{0}_V \rightarrow \vec{0}_W$

p.344

not in book!

Read Ex 10

$\underline{T(\vec{x}) = \overset{m\times n}{A}\vec{x}}$

$\mathbb{R}^n \rightarrow \mathbb{R}^m$

$\vec{x} \rightarrow A\vec{x}$

"T is 1-1" means "$A\vec{x} = \vec{b}$ has 0 or 1 sol'ns for every $\vec{b}$ in $\mathbb{R}^m$"

$\Updownarrow$ p.344

"$\mathrm{Ker}(T) = \{\vec{0}\}$" means "$A\vec{x} = \vec{0}$ has only the $\vec{0}$ sol'n" $\quad$ only $\vec{0}_{\mathbb{R}^n} \longrightarrow \vec{0}_{\mathbb{R}^m}$

$\Updownarrow$

Each col. (LI) of a row-ech. shape of A has a PP

Ex $[A] \underset{\text{shape}}{\overset{\text{Row-}}{\underset{\text{ech.}}{\sim}}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ $\qquad [A|\vec{0}] \sim \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \begin{array}{l} \rightarrow x_1 = 0 \\ \rightarrow x_2 = 0 \\ \\ \end{array}$

$\left.\begin{array}{l} T: \mathbb{R}^2 \rightarrow \mathbb{R}^3 \\ T(\vec{x}) = A\vec{x} \end{array}\right\}$ is 1-1 $\qquad$ (rank = 2)

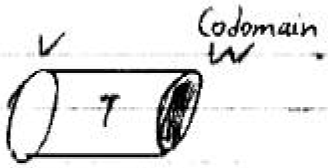
$$\boxed{\text{A LT is 1-1} \Leftrightarrow \underset{(\#\text{pivot cols.} = \#\text{cols.})}{\mathrm{rank}(T) = n}}$$

Why?

$$\mathrm{rank}(T) + \underbrace{\mathrm{nullity}(T)}_{\dim(\mathrm{Ker}(T))} = n$$

$= 0 \Leftrightarrow T$ is 1-1

(E) When is T "Onto"?



$\leftrightarrow$ each $\vec{w}$ in W has a nonempty preimage $\rightarrow \vec{w}$

Range = Codomain

$\leftrightarrow$ Range$(T) = W$

$\leftrightarrow$ $\underbrace{\text{rank}(T)}_{=\text{dim of Range}} = \dim(W)$

If $T(\vec{x}) = A\vec{x}$

T is onto $\Leftrightarrow$ $A\vec{x} = \vec{b}$ is consistent for every $\vec{b}$ in $\mathbb{R}^m$

$\Leftrightarrow$ A's cols. span $\mathbb{R}^m$ $\quad \mathbb{R}^n \quad \mathbb{R}^m$ "$\vec{x}$'s" $\exists$ all $\vec{b}$'s are hit

not in book

$\Leftrightarrow$ Each row of a row-ech. shape of A has a PP

— is what?

Ex $T: \mathbb{R}^3 \rightarrow \mathbb{R}^{2}$ dim(W)

$T(\vec{x}) = A\vec{x}$

$[A] \xrightarrow{\text{row-ech. shape}} \begin{bmatrix} 2 & 1 & 4 \\ 0 & 3 & 5 \end{bmatrix}$ } each row has a PP

T is onto

(F) What if $\dim(V) = \dim(W)$?

$\rightarrow$ A linear $T: V \rightarrow W$

(1) is one-to-one
(2) is onto

} is either **both** or **neither**

($\approx$ basis)

<u>Ex</u> $T: \mathbb{R}^3 \to \mathbb{R}^3$
$T(\vec{x}) = A\vec{x}$

$$[A] \sim \begin{bmatrix} 2 & -2 & 1 \\ 0 & 3 & 5 \\ 0 & 0 & 1 \end{bmatrix}$$

We have a full "set" of PPs.
(down the main diagonal)
so T is 1-1 and onto. T is an <u>isomorphism</u>.

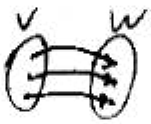
<u>Read Ex 11</u>

, webster
iso-equal, homog., uniform
morph-form
Gr-intellectuals
Romans-engineers, lawyers

<u>G Isomorphisms of Vector Spaces</u>

V and W are <u>isomorphic</u> if there is a one-to-one and onto linear $T: V \to W$ that "marries" the vectors of V with the vectors of W.

no bigamy



$\Leftrightarrow \dim(V) = \dim(W)$ (if $< \infty$)

Me:
You can have ∞-dim V.S.'s that are isom., but that story's more complicated

<u>Ex</u> $\mathbb{R}^3 \cong M_{3,1} \cong M_{1,3} \cong P_2 \cong$ a 3-D subspace of $\mathbb{R}^4$

a basis in your hot lil hands could be OG, ON

"Natural" correspondences $\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \leftrightarrow \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \leftrightarrow [c_1 \ c_2 \ c_3] \leftrightarrow c_1 + c_2x + c_3x^2 \leftrightarrow$ $\left\{ \begin{array}{l} \text{Basis}: \{\vec{b}_1, \vec{b}_2, \vec{b}_3\} \\ T: \text{This space} \to \mathbb{R}^3 \\ T(\underbrace{c_1\vec{b}_1 + c_2\vec{b}_2 + c_3\vec{b}_3}_{\text{a vector in the space}}) = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \end{array} \right.$

coords in $\mathbb{R}^3$

or $c_3 + c_2x + c_1x^2$

Adding vectors in $\mathbb{R}^3$ is equiv. to adding polys. in P_2!

These spaces "act the same" wrt (with respect to) "vector +", "scalar mult."

<u>Read Ex 12</u>

6.3: MATRICES FOR LTs

Ⓐ The Standard Basis for $\mathbb{R}^n$

$$\{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$$

$$\begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix} \quad \begin{bmatrix}0\\1\\0\\\vdots\\0\end{bmatrix} \quad \cdots \quad \begin{bmatrix}0\\0\\\vdots\\1\end{bmatrix}$$

If we know how a linear T maps a basis of $\mathbb{R}^n$, we know how T maps everything in $\mathbb{R}^n$.

Ⓑ The Standard Matrix for $T: \mathbb{R}^n \to \mathbb{R}^m$ (linear)

$$A = \begin{array}{c c} & \begin{array}{ccccc} 1 & & & & n \end{array} \\ \begin{array}{c} 1 \\ m \end{array} & \begin{bmatrix} T(\vec{e}_1) & T(\vec{e}_2) & \cdots & T(\vec{e}_n) \end{bmatrix} \end{array}$$

Column i is $T(\vec{e}_i)$, the image of $\vec{e}_i$.
(the i^{th} st. basis vector)

Then, $T(\vec{v}) = A\vec{v}$ for every $\vec{v}$ in $\mathbb{R}^n$.

<u>Why?</u>

I'll often switch between row vecs, col vecs - ok here.

$$T(\vec{v}) = T\left(\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}\right)$$

$$= T(v_1\vec{e}_1 + \dots + v_n\vec{e}_n)$$

$$= v_1 T(\vec{e}_1) + \dots + v_n T(\vec{e}_n)$$

$$A\vec{v} = [T(\vec{e}_1) \ \cdots \ T(\vec{e}_n)] \underbrace{\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}}_{\text{weights}}$$

$$\overset{\text{Trust me}}{=} v_1 T(\vec{e}_1) + \dots + v_n T(\vec{e}_n)$$

same

© Examples

<u>Ex</u> $T: \mathbb{R}^2 \to \mathbb{R}^3$

$T(x,y) = (x+y,\ 2x,\ 3x-4y)$

<u>Method 1</u> (Know!)

$$T(\vec{e}_1) = T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = (1+0,\ 2(1),\ 3(1)-4(0)) = (1,\ 2,\ 3)$$

$$T(\vec{e}_2) = T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = (0+1,\ 2(0),\ 3(0)-4(1)) = (1,\ 0,\ -4)$$

$$A = [T(\vec{e}_1)\ T(\vec{e}_2)] = \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ 3 & -4 \end{bmatrix}$$

Method 2 (Shortcut)

Write components of $T(x,y)$ in rows
(line up like terms)

$$\begin{array}{l} x + y \\ 2x \\ 3x - 4y \end{array}$$

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ 3 & -4 \end{bmatrix}$$

Ex 1 (p.351) Shortcut "Warning"

$T: \mathbb{R}^3 \to \mathbb{R}^2$
$T(x,y,z) = (x - 2y,\ 2x + y)$

$$\begin{array}{l} x - 2y + 0z \\ 2x + y + 0z \end{array}$$
Don't forget z column

$$A = \begin{bmatrix} 1 & -2 & 0 \\ 2 & 1 & 0 \end{bmatrix}$$

projection on x-axis in $\mathbb{R}^2$

Read Ex 2 (p.351-2)

Ex $T: \mathbb{R}^2 \to \mathbb{R}^2$, T stretches vectors by a factor of 2.

Method 1 $T(\vec{e}_1) = T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$
$T(\vec{e}_2) = T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$
$A = [T(\vec{e}_1)\ \ T(\vec{e}_2)] = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$
e.g., $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$

Method 2 $T(x,y) = (2x, 2y)$
etc.

up to 19

① Extensions

Composition: (Motivates the way we multiply matrices!)

Meyer 3.4
This is why we mult. matrices the way we do.
1855 - Arthur Cayley made this connection w/ comp. of linear funcs.
1857 - A Memoir on the Theory of Matrices - birth of modern linear alg.
(We already knew about dets, Gaussian elim.)

$$T_2 \circ T_1 (\vec{v}) = T_2(\overset{\text{linear}}{T_1(\vec{v})}) = A_2 A_1 \vec{v}$$

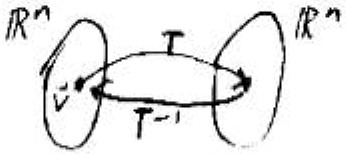
$$\underbrace{\underset{\substack{\uparrow \\ \text{apply 2nd} \\ \mathbb{R}^m \to \mathbb{R}^p}}{T_2} \circ \underset{\substack{\uparrow \\ \text{apply 1st} \\ \mathbb{R}^n \to \mathbb{R}^m}}{T_1}}_{\mathbb{R}^n \to \mathbb{R}^p} \qquad \underbrace{\underset{p \times m}{A_2} \; \underset{m \times n}{A_1}}_{\substack{\text{standard matrices} \\ p \times n}}$$

Read Ex 3 (p. 353)

Inverse:

$$T^{-1}(\vec{v}) = A^{-1}\vec{v}$$

T^{-1} ← undoes T



as opposed to nonlinear? ex ∡

Linear T is invertible $\Leftrightarrow$ A is invertible
$\updownarrow$
T is one-to-one (1-1) and onto (an isomorphism)

Read Ex 4 (pp. 354-5)

6.4: TRANSITION MATRICES and SIMILARITY

(A) Similar Matrices

A, A', A'' - $n \times n$

A' is <u>similar</u> to A $(A' \simeq A) \leftrightarrow$
There exists an <u>invertible</u> $n \times n$ matrix P
such that $A' = P^{-1}AP$

A few took 245

Book uses B, C
I need for Basis

Props. Similarity is an <u>equivalence relation</u>. Math 245!

① (A is similar to itself.) (Reflexivity) $A \simeq A$

What could I use for P?

Proof $A = I^{-1}AI$ (Let $P = I$)

② (If A' is similar to A, then A is similar to A'.) (Symmetry) If $A' \simeq A$ then $A \simeq A'$

Don't ' ' confuse w/ A'

Proof $A' \simeq A$

$\rightarrow$ there exists P: $A' = P^{-1}AP$

$$PA'P^{-1} = \underbrace{PP^{-1}}_{I}A\underbrace{PP^{-1}}_{I}$$

$$PA'P^{-1} = A$$

$$A = PA'P^{-1}$$

$$A = Q^{-1}A'Q \quad (\text{Let } Q = P^{-1})$$

③ (If A' is similar to A, and A'' is similar to A' then A'' is similar to A.) (Transitivity) If $A' \simeq A$ and $A'' \simeq A'$ then $A'' \simeq A$

Proof #19

$M_{1,1}$ classes $\{[c] \mid c \in \mathbb{R}\}$

$M_{n,n}$

$M_{n,n}$ can be partitioned into similarity classes.

(B) Transition Matrices

I figured you had plenty to do in Ch. 4 anyway

(From 4.7)

Let $B' = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ be a basis for $\mathbb{R}^n$.

Let $B = \{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$ be the standard basis.

The <u>transition matrix</u> from B' to B is

$$P = [\vec{v}_1 \;\; \vec{v}_2 \;\; \cdots \;\; \vec{v}_n]$$

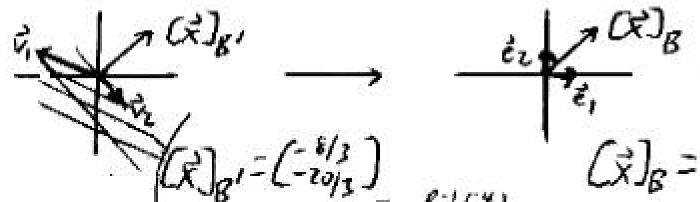
<u>Idea</u> If $[\vec{x}]_{B'}$ and $[\vec{x}]_B$ are the coords of $\vec{x}$ relative to B', B, then

$$\underbrace{P}_{B' \to B}[\vec{x}]_{B'} = [\vec{x}]_B$$

$$\rightarrow \quad [\vec{x}]_{B'} = \underbrace{P^{-1}}_{\substack{\text{transition} \\ \text{matrix} \\ B \to B'}}[\vec{x}]_B$$

<u>Ex</u> $B' = \{(-4,1), (1,-1)\}$
$B = \{(1,0), (0,1)\}$

$$P = \begin{bmatrix} -4 & 1 \\ 1 & -1 \end{bmatrix}$$



$[\vec{x}]_{B'} = \begin{bmatrix} -8/3 \\ -20/3 \end{bmatrix}$ $\xleftarrow{P^{-1}\begin{bmatrix} 4 \\ 4 \end{bmatrix}}$ $[\vec{x}]_B = P[\vec{x}]_{B'} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$

© The Matrix for T Relative to a Nonstandard Basis

Ex (#2)

a) Find the matrix A' for $T: \mathbb{R}^2 \to \mathbb{R}^2$
$T(x,y) = (x+y, 4y)$
relative to the basis $B' = \{(-4,1), (1,-1)\}$.

same B' as before.

Solution

The standard matrix for T
(relative to the standard basis) is:

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 4 \end{bmatrix} \begin{matrix} x+y \\ 4y \end{matrix}$$

The transition matrix from B' to B is:

$$P = \begin{bmatrix} -4 & 1 \\ 1 & -1 \end{bmatrix}$$

Then, the transition matrix from B to B' is:

$$P^{-1} = \frac{1}{\underbrace{\det(P)}_{3}} \begin{bmatrix} -1 & -1 \\ -1 & -4 \end{bmatrix}$$

$$= -\tfrac{1}{3} \begin{bmatrix} 1 & 1 \\ 1 & 4 \end{bmatrix}$$

$$\underset{\substack{T \\ \text{rel. to} \\ B'}}{A'} = \underset{\substack{③ \\ B \to B'}}{P^{-1}}\ \underset{\substack{② \, T \\ \text{rel. to} \\ B}}{A}\ \underset{① \, B' \to B}{P}$$

A' operates entirely within the context of the new, nonstandard (?) coord. system.

Matrix mult. is assoc.

$$A' = -\frac{1}{3}\begin{bmatrix} 1 & 1 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} -4 & 1 \\ 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{3} & \frac{4}{3} \\ -\frac{13}{3} & \frac{16}{3} \end{bmatrix}$$

Describes T relative to B'.

rel. to this coord system.

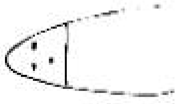
± Meyer 256

"maybe" because $A \sim A$

Two matrices represent the same linear $T: V \to V$, maybe w/respect to different bases $\Longleftrightarrow$ similar.

Bases: $B' \leftrightarrow B \leftrightarrow B''$ etc.

Matrices: for T $A' \leftrightarrow A \leftrightarrow A''$ etc.



Anton 8[ed] p.406
S.m. Invariants
Det, Rank, Nullity,
Trace, Evals,
Char poly, Espace dim
(not evecs)

What do these similar matrices have in common?

in HW

<u>Similarity invariants</u>

trace (Σa_{ii})

det

What's arguably most important # associated w/ $n \times n$ matrix?

rank, nullity

eigenvalues (Ch. 7)

Meyer 256

Quote eloquently

Brute force, crank thru some #s

→ Conjecture

→ Pf (by induction!)

Theory goal: Isolate props. of LT and their matrices that are coord- (basis-) independent.

6.4.6

From Matrix Analysis and Applied Linear Algebra by Carl Meyer

We share a philosophy about linear algebra: we think basis-free,
but when the chips are down we close the office door
and compute with matrices like fury.
Irving Kaplansky (1917–) *speaking about Paul Halmos* (1916–)