

Put some dynamism in L1.
 (not that it wasn't there before...)

CH. 6: LINEAR TRANSFORMATIONS

6.1: INTRO

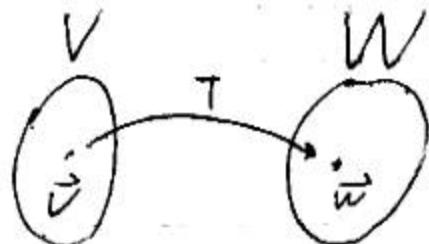
Ⓐ Terminology

W doesn't have
to be subspace

Let V, W be VS

$T: V \rightarrow W$ means:

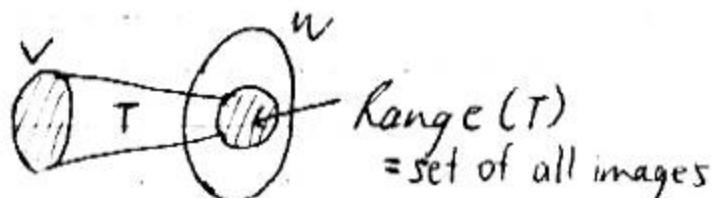
The function T maps the domain V into the codomain W .



$T(\vec{v}) = \vec{w}$
 ↑ "the image of \vec{v}
 in the preimage of \vec{w}

maybe $\vec{v} \rightarrow \vec{w}$
 no one $\rightarrow \vec{w}$

Range of $T = \{T(\vec{v}) \mid \vec{v} \text{ is in } V\}$



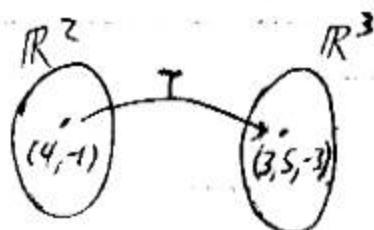
B) Ex

For any $\vec{v} = (v_1, v_2)$ in \mathbb{R}^2 , let $T: \mathbb{R}^2 \rightarrow \mathbb{R}^3$
be defined by $T(v_1, v_2) = (v_1 + v_2, v_1 - v_2, 3v_2)$

a) Find the image of $\vec{v} = (4, -1)$

$$\begin{aligned} T(\vec{v}) &= T(4, -1) \\ &= (4 + (-1), 4 - (-1), 3(-1)) \\ &= (3, 5, -3) \end{aligned}$$

$$\begin{aligned} \text{Only } (4, -1) &\rightarrow \vec{w} \\ 3v_2 = -3 &\rightarrow v_2 = -1 \\ v_1 + v_2 = 3 &\rightarrow v_1 = 4 \end{aligned}$$



b) Find the preimage of $\vec{w} = (4, 2, 3)$

Find all \vec{v} such that

$$\begin{aligned} T(\vec{v}) &= \vec{w} \\ (v_1 + v_2, v_1 - v_2, 3v_2) &= (4, 2, 3) \end{aligned}$$

Here, system
easy.

$$\begin{cases} v_1 + v_2 = 4 \\ v_1 - v_2 = 2 \\ 3v_2 = 3 \end{cases}$$

$$\underbrace{\left[\begin{array}{cc|c} 1 & 1 & 4 \\ 1 & -1 & 2 \\ 0 & 3 & 3 \end{array} \right]}_A$$

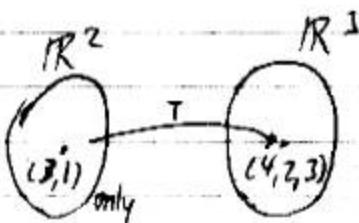
6.1.3

We get the unique sol'n

$$v_1 = 3$$

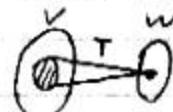
$$v_2 = 1$$

$$\{(3, 1)\}$$



Up to 5

If ∞ many sol'n's \rightarrow
parametrize sol'n set



③ Linear Transformations (LTs)

$T: V \rightarrow W$ is a LT if, for all \vec{v}_1, \vec{v}_2 in V and any scalar c ,

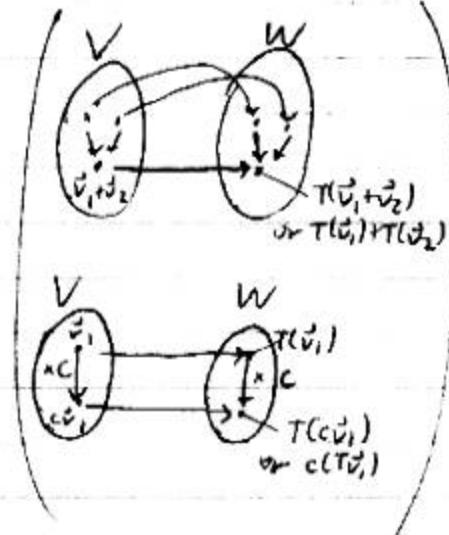
Image of sum
= sum of images

can add,
then map
or map,
then add
 \rightarrow same result

You can
reverse before
or after T
 \rightarrow applied \rightarrow
same result.

$$\textcircled{1} \quad T(\vec{v}_1 + \vec{v}_2) = \underbrace{T(\vec{v}_1)}_{\substack{\text{add in } V, \text{ 1st} \\ \text{then, map to } W}} + \underbrace{T(\vec{v}_2)}_{\substack{\text{map to } W, \text{ 1st} \\ \text{then, add in } W}}$$

$$\textcircled{2} \quad T(c\vec{v}_1) = c \underbrace{T(\vec{v}_1)}_{\substack{\text{map to } W, \text{ 1st} \\ \text{then, } \times c}}$$



Key Prop.

$$T(\vec{0}) = \vec{0}_W$$

$$\textcircled{3} \quad T(c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_n\vec{v}_n) = c_1T(\vec{v}_1) + \dots + c_nT(\vec{v}_n)$$

\vec{v}_i 's are in V
This leads to...

⑦ LTs and Bases

If we know $T(b_1), T(b_2), \dots, T(b_n)$ for any one basis $\{b_1, b_2, \dots, b_n\}$ of V , then we can find $T(v)$ for any v in V .

If we know how T maps a basis of V , we know how T maps everything in V .

Ex Let $T: \mathbb{R}^2 \rightarrow \mathbb{R}^3$ be a LT such that

$$\begin{aligned} T(1, 0) &= (1, 3, -2) \\ T(0, 1) &= (4, 2, 0) \end{aligned}$$

Find $T(-3, 2)$.

$$\begin{aligned} T\left(\begin{bmatrix} -3 \\ 2 \end{bmatrix}\right) &= T\left(\begin{bmatrix} -3 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix}\right) \\ &= T\left(-3\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) \\ &\stackrel{*}{=} -3T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) + 2T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) \\ &= -3(1, 3, -2) + 2(4, 2, 0) \\ &= (-3, -9, 6) + (8, 4, 0) \\ &= \boxed{(5, -5, 6)} \end{aligned}$$

You can find any $T(v_1, v_2)$

You don't have
to do this
I'll be switching
between
row vectors
etc - if
often doesn't
matter in this
material here.

E) The LT Given by a Matrix

$T(\vec{v}) = A\vec{v}$ defines a LT

$$T: \mathbb{R}^n \rightarrow \mathbb{R}^m$$

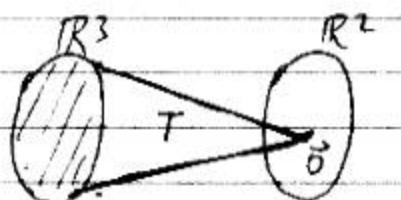
$$m \begin{bmatrix} & & n \\ A & & \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} w_1 \\ \vdots \\ w_m \end{bmatrix}$$

$\vec{v} \text{ in } \mathbb{R}^n \quad T(\vec{v}) \text{ in } \mathbb{R}^m$
(column vectors)

Ex If $A = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}_{2 \times 3}$, then

$T(\vec{v}) = A\vec{v}$ defines the zero transformation

$$T: \mathbb{R}^3 \rightarrow \mathbb{R}^2$$

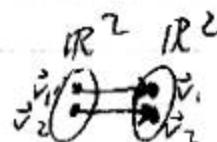


Ex If $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, then

$T(\vec{v}) = A\vec{v}$ defines the identity transformation

$$T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$$

Here, $T(\vec{v}) = \vec{v}$ for all \vec{v} in $V = W = \mathbb{R}^2$



(Last)

Ex If $A = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$, then

(see Section 12.4, Ch II, Swokowski)

$$T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$$

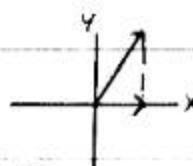
rotates every vector
in \mathbb{R}^2 counterclockwise about the
angle origin.

(Optional proof - p. 332)

Ex If $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, then

$T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is a projection in \mathbb{R}^2 .

$$T(\vec{v}) = A\vec{v} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ 0 \end{bmatrix}$$



Ex 8 - p. 332: \mathbb{R}^3

6.1.7

Ex Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

Let $T(\vec{v}) = A\vec{v}$ define $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$

a) Find the image of $(4, 3)$.

$$T(4, 3) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

b) Find the preimage of $(3, 0)$

$$\begin{aligned} T(\vec{v}) &= A\vec{v} \\ A\vec{v} &= T(\vec{v}) \end{aligned}$$

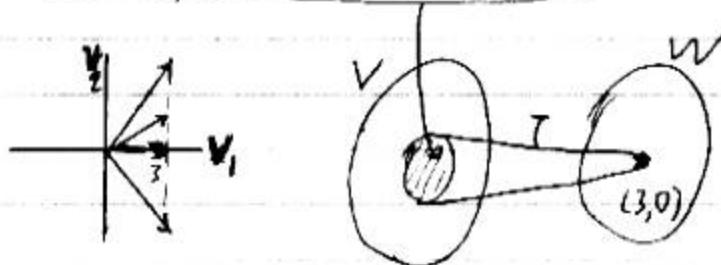
$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

$$\left(\begin{bmatrix} 1 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix} \right)$$

$$v_1 = 3$$

$$v_2 = t \text{ (free)}$$

$\{(3, t) | t \text{ is a real #}\}$



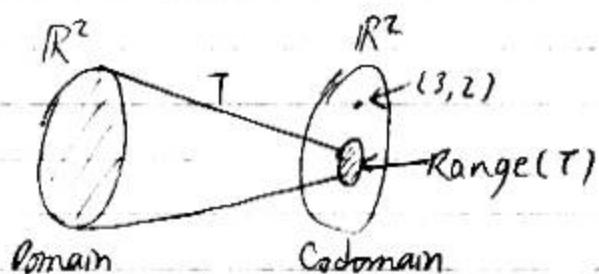
6.1.8

c) Find the preimage of $(3, 2)$

$$\begin{bmatrix} 1 & 0 & | & 3 \\ 0 & 0 & | & 2 \end{bmatrix}$$

Preimage = \emptyset

He doesn't get
to play.
He's left out.



Go back to rotation Ex
(6.1.6)

6.2: KERNEL + RANGE

Let $T: V \rightarrow W$ be a L.T.

(A) $\text{Ker}(T)$

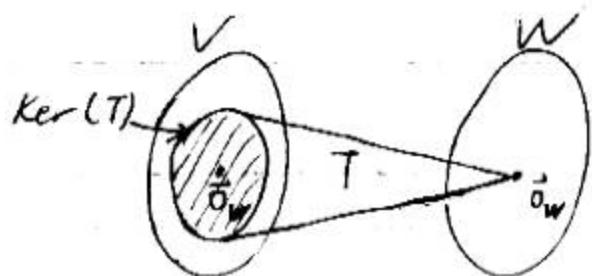
= kernel of T

$$= \{\vec{v} \in V \mid T(\vec{v}) = \vec{0}_w\}$$

= set of vectors in V that T maps to $\vec{0}$ in W .

is a subspace of V (pp. 339-40)

What vector
in V must be
in the kernel?
subspace



Ex $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$, find $\text{Ker}(T)$ if
 $T(x, y) = (\underline{0}, y)$ ← projection on y-axis
 what makes
 this = \emptyset ?

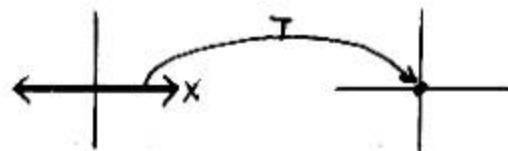
6.2.2

$$\begin{cases} \cancel{0 = 0} \\ y = 0 \end{cases}$$

Your book
sticker w/x

$$\begin{cases} x = t & (\text{free}) \\ y = 0 \end{cases}$$

$$\boxed{\begin{aligned} \text{Ker}(T) &= \{(t, 0) \mid t \text{ is a real \#}\} \\ &= \text{x-axis} \end{aligned}}$$



$T(\vec{x}) = A\vec{x}^{m \times n}$ defines $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ (A LT can be given by a matrix.)

$$\begin{aligned} \text{Ker}(T) &= \{\vec{x} \text{ in } \mathbb{R}^n \mid A\vec{x} = \vec{0}\} \\ &= N(A), \text{ the } \underline{\text{nullspace}} \text{ of } A \end{aligned}$$

Its dimension = nullity(T)
or nullity(A)

Ex $T: \mathbb{R}^3 \rightarrow \mathbb{R}^2$

$$T(\vec{x}) = A\vec{x}, \vec{x} \text{ in } \mathbb{R}^3$$

$$A = \begin{bmatrix} 2 & 4 & 6 \\ 0 & 1 & 3 \end{bmatrix}$$

Find a basis for $\ker(T)$ as a subspace of \mathbb{R}^3 .

Solution

Solve $A\vec{x} = \vec{0}$ and find a basis for $N(A)$.

$$\left[\begin{array}{ccc|c} 2 & 4 & 6 & 0 \\ 0 & 1 & 3 & 0 \end{array} \right]$$

$$\xrightarrow{\text{RREF}} \left[\begin{array}{ccc|c} 1 & 0 & -3 & 0 \\ 0 & 1 & 3 & 0 \end{array} \right]$$

$$\left\{ \begin{array}{l} x_1 - 3x_3 = 0 \\ x_2 + 3x_3 = 0 \end{array} \right. \quad \begin{array}{l} x_3 \\ \text{free} \end{array}$$

$$\left\{ \begin{array}{l} x_1 = 3x_3 \\ x_2 = -3x_3 \end{array} \right.$$

$$x_3 = t$$

$$\left\{ \begin{array}{l} x_1 = 3t \\ x_2 = -3t \\ x_3 = t \end{array} \right.$$

6.2.4

$$\vec{x} = t \begin{bmatrix} 3 \\ -3 \\ 1 \end{bmatrix}, t \text{ is any real #} \quad \left. \right\} \begin{array}{l} \text{Ker}(T): \\ \text{line in } \mathbb{R}^3 \end{array}$$

Basis: $\left\{ \begin{bmatrix} 3 \\ -3 \\ 1 \end{bmatrix} \right\}$

$$\text{nullity}(T) = 1$$

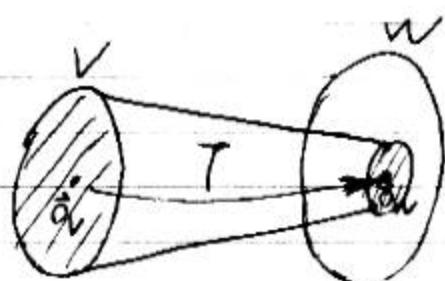
up to 6

Read Exs 1-6

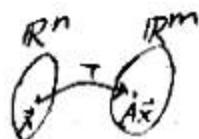
(B) Range(T)

$$= \{ T(\vec{v}) \mid \vec{v} \text{ is in } V \} \text{ (images)}$$

is a subspace of W



$$\underline{T(\vec{x}) = A\vec{x}^{m \times n}}$$



Then, Range(T) = $\{A\vec{x} \mid \vec{x} \text{ is in } R^n\}$

= $\{b \text{ in } R^m \mid A\vec{x} = b \text{ is consistent}\}$

= Col(A), the column space of A

Its dimension = rank(T) or rank(A)

Ex $T: R^4 \rightarrow R^3$

$$T(\vec{x}) = A\vec{x}, \vec{x} \text{ in } R^4$$

$$A = \begin{bmatrix} (1) & 2 & (4) & 3 \\ 1 & 2 & 5 & 5 \\ 2 & 4 & 8 & 6 \end{bmatrix}$$

Find a basis for Range(T) as a subspace of R^3 .

Solution

→ row-echelon "shape"

$$A \sim \begin{bmatrix} (1) & 2 & 4 & 3 \\ 0 & 0 & (1) & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

↑ ↑
cols 1,3 are pivot cols.

Take cols. 1, 3 from A .

$$\text{Basis} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 8 \end{bmatrix} \right\}$$

Up to 11
Also, $\text{Range}(T) = \text{Span}\{\vec{v}_1, \vec{v}_2\} = \text{"plane in } \mathbb{R}^3"$
 $\text{rank}(T) = 2$

② Sum Formulas

$$\textcircled{4.1f} \quad \text{rank}(T) + \text{nullity}(T) = n$$

$$\dim(\text{Range}(T)) + \dim(\text{Ker}(T)) = \dim(\text{domain}, V)$$

$$T(\vec{x}) = A\vec{x}$$

$$[A] \sim \begin{bmatrix} \text{shape} \\ \text{row-each} \end{bmatrix}.$$

$$\text{rank}(T) = \# \text{pivot cols}$$

$$\text{nullity}(T) = \# \text{nonpivot cols}$$

Read Exs. 8, 9

Ex $T: \mathbb{R}^3 \rightarrow \mathbb{R}^2$
 $T(\vec{x}) = A\vec{x}$

from
 $\text{Ker}(T)$

$$A \sim \begin{bmatrix} 2 & 4 & 6 \\ 0 & 1 & 3 \end{bmatrix}$$

$$\left(\sim \begin{bmatrix} 1 & 8 & -3 \\ 0 & 1 & 3 \end{bmatrix} \right)$$

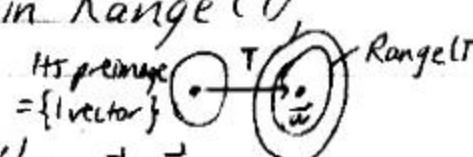
Be aware of
variations of
same?

Up to 35

$$\text{rank}(T) = \dim(\text{Range}(T)) = 2$$

$$\begin{aligned} \text{nullity}(T) &= \dim(\text{Ker}(T)) = n - \text{rank}(T) \\ &= 3 - 2 \\ &= 1 \end{aligned}$$

① When is T "One-to-One" (1-1)?

\Leftrightarrow For each \vec{w} in Range(T)


\Leftrightarrow whenever $T(\vec{v}_1) = T(\vec{v}_2)$, then $\vec{v}_1 = \vec{v}_2$



NO:

$\vec{v}_1 \rightarrow$, can't have 2
 $\vec{v}_2 \rightarrow$, vectors w/
same image

6.2.8

A LT is 1-1 $\Leftrightarrow \text{Ker}(T) = \{\vec{0}_m\}$ Proof optional p.344

p.344

not in book!

$$T(\vec{x}) = A\vec{x} \quad \begin{matrix} \text{m} \\ \times \\ n \end{matrix}$$

Read Ex 10only $\vec{0}_n \rightarrow \vec{0}_m$

$$\begin{matrix} \mathbb{R}^n & \xrightarrow{\quad} & \mathbb{R}^m \\ \vec{x} & \rightarrow & \vec{A}\vec{x} \end{matrix}$$

" T is 1-1" means " $A\vec{x} = \vec{b}$ has 0 or 1 solns for every \vec{b} in \mathbb{R}^m ". \vec{b} has 0 solns in $\text{Range}(T)$ "

" $\text{Ker}(T) = \{\vec{0}\}$ " means " $A\vec{x} = \vec{0}$ has only the $\vec{0}$ sol'n"

Each col. of a non-ech. shape of A has a (PP)

$$\text{Ex } [A] \begin{matrix} \text{Row-} \\ \text{ech.} \\ \text{shape} \end{matrix} \sim \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right]$$

$$[A|\vec{0}] \sim \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right] \Rightarrow \begin{array}{l} x_1 = 0 \\ x_2 = 0 \end{array}$$

$T: \mathbb{R}^2 \rightarrow \mathbb{R}^3$ } is 1-1
 $T(\vec{x}) = A\vec{x}$ (rank $= 2$)

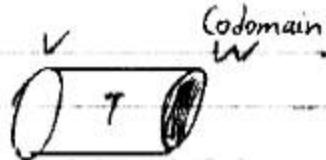
A LT is 1-1 $\Leftrightarrow \text{rank}(T) = n$
 $(\# \text{pivot cols.} = \# \text{cols.})$

Why?

$$\text{rank}(T) + \underbrace{\text{nullity}(T)}_{\dim(\text{Ker}(T))} = n$$

 $= 0 \Leftrightarrow T$ is 1-1

E) When is T "onto"?



\Leftrightarrow each w in W has a nonempty preimage \rightarrow

$\stackrel{\text{Range}}{=}$ $\stackrel{\text{Codomain}}{=}$

$$\Leftrightarrow \text{Range}(T) = W$$

$$\Leftrightarrow \underset{=\dim \text{Range}}{\text{rank}(T)} = \dim(W)$$

If $T(\vec{x}) = A\vec{x}$

- T is onto $\Leftrightarrow A\vec{x} = \vec{b}$ is consistent for every \vec{b} in \mathbb{R}^m
- $\Leftrightarrow A$'s cols. span \mathbb{R}^m \mathbb{R}^m all \vec{b} 's are hit
- \Leftrightarrow Each row of a row-ech. \vec{x}_s \vec{b} shape of A has a (PP)

Ex $T: \mathbb{R}^3 \rightarrow \mathbb{R}^2$ $\dim(W)$
 $T(\vec{x}) = A\vec{x}$

$$(A) \begin{matrix} \text{row} \\ \text{ech} \\ \text{shape} \end{matrix} \left[\begin{matrix} 2 & 1 & 4 \\ 0 & 3 & 5 \end{matrix} \right] \quad \text{each row has a (PP)}$$

T is onto

F) What if $\dim(V) = \dim(W)$?

\Rightarrow A linear $T: V \rightarrow W$ is

- (1) is one-to-one \Leftrightarrow either both
- (2) is onto \Leftrightarrow or neither
 $(\approx$ basis)

Ex $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$
 $T(\mathbf{x}) = A\mathbf{x}$

$$[A] \sim \begin{bmatrix} 2 & -3 & 1 \\ 0 & 3 & 5 \\ 0 & 0 & 1 \end{bmatrix}$$

We have a full "set" of PPs.

(down the main diagonal)

so T is 1-1 and onto. T is an isomorphism.

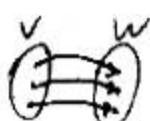
Read Ex 11

Webster
 (50 equal, homog.,
 uniform
 morphism
 Gr.-thinkerly
 engineer-engineer,
 tangent

no bigamy

⑥ Isomorphisms of Vector Spaces

V and W are isomorphic if there is a one-to-one and onto linear $T: V \rightarrow W$ that "marries" the vectors of V with the vectors of W .



$\Leftrightarrow \dim(V) = \dim(W)$ (if $<\infty$)

Ex $\mathbb{R}^3 \cong M_{3,1} \cong M_{1,3} \cong P_2 \cong$ a 3-D subspace of \mathbb{R}^4

"Natural" correspondences $\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \leftrightarrow [c_1 \ c_2 \ c_3] \leftrightarrow c_1 + c_2x + c_3x^2 \leftrightarrow \left\{ \begin{array}{l} \text{Basis: } \{\vec{b}_1, \vec{b}_2, \vec{b}_3\} \\ T: \text{This space} \rightarrow \mathbb{R}^3 \\ T(c_1\vec{b}_1 + c_2\vec{b}_2 + c_3\vec{b}_3) = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \end{array} \right.$

Adding vectors in \mathbb{R}^3 is equiv. to adding polys. in P_2 .

These spaces "act the same" wrt (with respect to)
 "vector +", "scalar mult."

Read Ex 12

6.3: MATRICES FOR LTs(A) The Standard Basis for \mathbb{R}^n

$$\{\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n\}$$

$\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$
 $\begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}$
 $\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$

If we know how a linear T maps a basis of \mathbb{R}^n , we know how T maps everything in \mathbb{R}^n .

(B) The Standard Matrix for $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ (linear)

$$A = [T(\vec{e}_1) \ T(\vec{e}_2) \ \dots \ T(\vec{e}_n)]$$

Column i is $T(\vec{e}_i)$, the image of \vec{e}_i .
(the i^{th} st. basis vector)

Then, $T(\vec{v}) = A\vec{v}$ for every \vec{v} in \mathbb{R}^n .

Why?

I'll often
switch between
row vector, col
vev - due here.

$$\begin{aligned}
 T(\vec{v}) &= T\left(\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}\right) \\
 &= T(v_1 \vec{e}_1 + \dots + v_n \vec{e}_n) \\
 &= v_1 T(\vec{e}_1) + \dots + v_n T(\vec{e}_n) \\
 A\vec{v} &= [T(\vec{e}_1) \ \dots \ T(\vec{e}_n)] \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \\
 &\stackrel{\text{most}}{=} v_1 T(\vec{e}_1) + \dots + v_n T(\vec{e}_n)
 \end{aligned}$$

weights

same

① Examples

Ex $T: \mathbb{R}^2 \rightarrow \mathbb{R}^3$
 $T(x, y) = (x+y, 2x, 3x-4y)$

Method 1 (Know!)

$$\begin{aligned}
 T(\vec{e}_1) &= T([1]) \\
 &= (1+0, 2(1), 3(1)-4(0)) \\
 &= (1, 2, 3)
 \end{aligned}$$

$$\begin{aligned}
 T(\vec{e}_2) &= T([0]) \\
 &= (0+1, 2(0), 3(0)-4(1)) \\
 &= (1, 0, -4)
 \end{aligned}$$

$$A = [T(\vec{e}_1) \ T(\vec{e}_2)] = \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ 3 & -4 \end{bmatrix}$$

Method 2 (Shortcut)

Write components of $T(x, y)$ in rows
(line up like terms)

$$\begin{matrix} x + y \\ 2x \\ 3x - 4y \end{matrix}$$

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ 3 & -4 \end{bmatrix}$$

Ex 1 (p. 351) Shortcut "Warning"

$$T: \mathbb{R}^3 \rightarrow \mathbb{R}^2$$

$$T(x, y, z) = (x - 2y, 2x + y)$$

$$\begin{matrix} x - 2y \\ 2x + y \end{matrix}$$

+ $\begin{matrix} 0 \\ z \end{matrix}$
Don't forget
z column

$$A = \begin{bmatrix} 1 & -2 & 0 \\ 2 & 1 & 0 \end{bmatrix}$$

projection
on x-axis
in \mathbb{R}^2

Read Ex 2 (p. 351-2)

Ex $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$, T stretches vectors by a factor of 2.

Method 1 $T(\vec{e}_1) = T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

$$T(\vec{e}_2) = T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

Method 2 $T(x, y) = (2x, 2y)$
etc.

$$A = [T(\vec{e}_1) \quad T(\vec{e}_2)] = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

e.g., $\begin{bmatrix} 3 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$

up to 19

① Extensions

Composition: (Motivates the way we multiply matrices!)

Meyer 3.4
 This is why we mult. matrices the way we do.
 1855 - Arthur Cayley made this connection w/ comp. of linear func's.
 1857 - A Memoir on the Theory of Matrices - birth of modern linear alg.
 (we already knew about dets, Gaussian elim.)

$$T_2 \circ T_1(\vec{v}) = T_2(T_1(\vec{v})) = A_2 A_1 \vec{v}$$

$\begin{matrix} \uparrow & \uparrow \\ \text{apply 2nd} & \text{apply 1st} \\ R^m \rightarrow R^p & R^n \rightarrow R^m \\ \hline R^n \rightarrow R^p \end{matrix}$

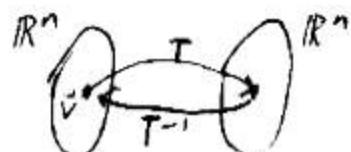
$\begin{matrix} / & | \\ p \times m & m \times n \\ \underbrace{\text{standard matrices}}_{p \times m} \end{matrix}$

Read Ex 3 (p. 353)

Inverse:

$$T^{-1}(\vec{v}) = A^{-1}\vec{v}$$

\nwarrow undoes T



as opposed to nonlinear?
 ex. $\not\propto$

Linear T is invertible $\Leftrightarrow A$ is invertible

\nwarrow
 T is one-to-one and onto
 (an isomorphism)



Read Ex 4 (pp. 354-5)

6.4: TRANSITION MATRICES and SIMILARITY

(A) Similar Matrices

$A, A', A'' - n \times n$

A' is similar to A ($A' \sim A$) \Leftrightarrow

There exists an invertible $n \times n$ matrix P
such that $A' = P^{-1}AP$

A few took
245

book uses B, C
I need for Banis

Props. Similarity is an equivalence relation. Math 245!

① (A' is similar to itself.) (Reflexivity)

$A \sim A$

Proof $A = I^{-1}AI$ (let $P=I$)

② (If A' is similar to A ,
then A is similar to A'). (Symmetry)

If $A' \sim A$
then $A \sim A'$

Proof $A' \sim A$
 \rightarrow there exists P : $A' = P^{-1}AP$

$$PA'P^{-1} = \underbrace{PP^{-1}}_{I} \underbrace{APP^{-1}}_{I}$$

$$PA'P^{-1} = A$$

$$A = PA'P^{-1}$$

$$A = Q^{-1}A'Q \quad (\text{let } Q = P^{-1})$$

③ (If A' is similar to A ,
and A'' is similar to A' ,
then A'' is similar to A). (Transitivity)

If $A' \sim A$
and $A'' \sim A'$
then $A'' \sim A$

Proof #19

$M_{n,n}$ classes
 $\{[c]\} \subset R^n$

$M_{n,n}$ can be partitioned into similarity classes.



$$M_{n,n}$$

1	2	3	4	5
---	---	---	---	---

I figured you
had plenty to
do in Ch. 4 anyway

(From 4.7)

Let $B' = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ be a basis for R^n .

Let $B = \{\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n\}$ be the standard basis.

The transition matrix from B' to B is

$$P = [\vec{v}_1 \ \vec{v}_2 \ \dots \ \vec{v}_n]$$

Idea If $[\vec{x}]_{B'}$ are the coords of \vec{x} relative to B' ,
 and $[\vec{x}]_B$
 then

$$\underset{B' \rightarrow B}{P} [\vec{x}]_{B'} = [\vec{x}]_B$$

$$\rightarrow [\vec{x}]_{B'} = \underbrace{P^{-1} [\vec{x}]_B}_{\substack{\text{transition} \\ \text{matrix} \\ B \rightarrow B'}}$$

$$\text{Ex } B' = \{(-4, 1), (1, -1)\} \\ B = \{(1, 0), (0, 1)\}$$

$$P = \begin{bmatrix} -4 & 1 \\ 1 & -1 \end{bmatrix}$$

$$[\vec{x}]_{B'} = \begin{bmatrix} -8/3 \\ -2/3 \end{bmatrix} \xrightarrow{P^{-1}} [\vec{x}]_B = P[\vec{x}]_{B'} = \begin{bmatrix} 4 \\ -4 \end{bmatrix}$$

⑥ The Matrix for T Relative to a Nonstandard Basis

Ex (#2)

a) find the matrix A' for $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$

same B'
as before.

$T(x, y) = (x+y, 4y)$
relative to the basis $B' = \{(-4, 1), (1, -1)\}$.

Solution

The standard matrix for T
(relative to the standard basis) is:

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 4 \end{bmatrix}$$

The transition matrix from B' to B is:

$$P = \begin{bmatrix} -4 & 1 \\ 1 & -1 \end{bmatrix}$$

Then, the transition matrix from B to B' is:

$$P^{-1} = \frac{1}{\det(P)} \begin{bmatrix} -1 & -1 \\ -1 & -4 \end{bmatrix}$$

$$= -\frac{1}{3} \begin{bmatrix} 1 & 1 \\ 1 & 4 \end{bmatrix}$$

$$A' = P^{-1}AP$$

↑
 T
 rel. to
 B'
 $B \rightarrow B'$
③
②
①
 T
 rel. to
 B
 $B' \rightarrow B$

A' operates entirely within the context of the new, nonstandard (?) coord. system.

Matrix mult.
is assoc.

$$A' = -\frac{1}{3} \begin{bmatrix} 1 & 1 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} -4 & 1 \\ 1 & -1 \end{bmatrix}$$

$$= \begin{pmatrix} -\frac{1}{3} & \frac{4}{3} \\ -\frac{13}{3} & \frac{16}{3} \end{pmatrix}$$

Describes T relative to B' .

rel. to this
coord system.



+ Meyer 256
"maybe" because
 $A \neq A'$

Two matrices represent the same linear
 $T: V \rightarrow V$, maybe w.r.t. different
bases \longleftrightarrow similar.

Bases:

```

    B'   B"
    ↗     ↘
      B   etc.
    
```

Matrices:
for T

```

    A'   A"
    ↗     ↘
      A   etc.
    
```



Anton 8 ed p.416
S.m. Invariants
Det, Rank, Nullity,
Trace, Evals,
Char poly, Space dim
(not evecs)

What do these similar matrices have in common?

Similarity invariants

trace ($\sum a_{ii}$)

det

rank, nullity

eigenvalues (Ch. 7)

in HW

What's arguably
most important
but omitted w/
mn matrix?

Meyer 256
Quote:
"unluckily
finite base, rank
then some #'s
→ Conjecture
→ PF (by later?)

Theory goal: Isolate props. of LT and their matrices
that are coord- (basis-) independent.