Putt some dynamism in L.A. (not that it wasn't there before...)

CH. 6: LINEAR TRANSFORMATIONS

6.1: INTRO

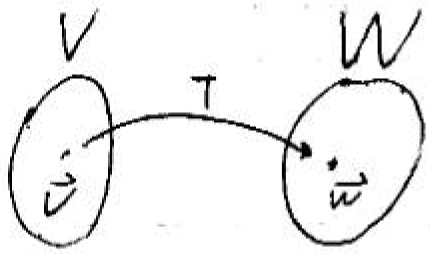
Ⓐ Terminology

W doesn't have to be subspace

Let V, W be VSs

$$T : V \longrightarrow W \quad \text{means:}$$

The function T that maps the domain V into the codomain W.



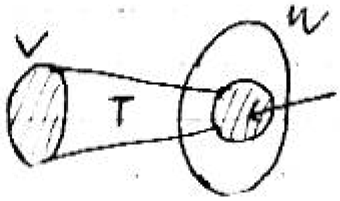
$T(\vec{v}) = \vec{w}$ ← the image of $\vec{v}$

↑ in the preimage of $\vec{w}$

maybe $\rightrightarrows \cdot \vec{w}$

no one $\to \vec{w}$

Range of $T = \{ T(\vec{v}) \mid \vec{v} \text{ is in } V \}$



Range(T) = set of all images

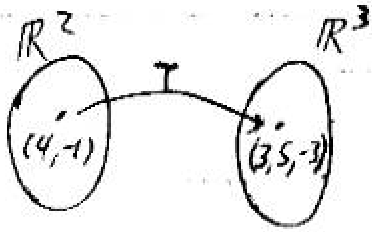
B) Ex

For any $\vec{v} = (v_1, v_2)$ in $\mathbb{R}^2$, let $T: \mathbb{R}^2 \to \mathbb{R}^3$ be defined by $T(v_1, v_2) = (v_1 + v_2,\ v_1 - v_2,\ 3v_2)$

a) Find the image of $\vec{v} = (4, -1)$

$$\begin{aligned} T(\vec{v}) &= T(4, -1) \\ &= (4 + (-1),\ 4 - (-1),\ 3(-1)) \\ &= (3, 5, -3) \end{aligned}$$

Only $(4,-1) \to \vec{w}$
$3v_2 = -3 \to v_2 = -1$
$v_1 + v_2 = 3 \to v_1 = 4$



b) Find the preimage of $\vec{w} = (4, 2, 3)$

Find all $\vec{v}$ such that

$$T(\vec{v}) = \vec{w}$$
$$(v_1 + v_2,\ v_1 - v_2,\ 3v_2) = (4, 2, 3)$$

Here, system easy.

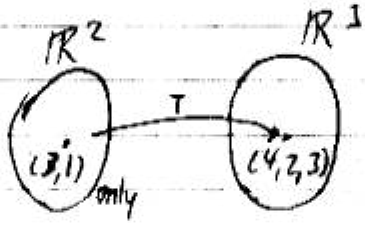
$$\begin{cases} v_1 + v_2 = 4 \\ v_1 - v_2 = 2 \\ \quad\ 3v_2 = 3 \end{cases}$$

$$\left[\begin{array}{cc|c} \underbrace{\begin{matrix} 1 & 1 \\ 1 & -1 \\ 0 & 3 \end{matrix}}_{A} & \begin{matrix} 4 \\ 2 \\ 3 \end{matrix} \end{array}\right]$$

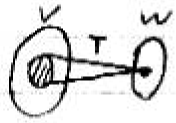
We get the unique sol'n
$v_1 = 3$
$v_2 = 1$

$\{(3,1)\}$

up to 5



If ∞ many sol'ns $\to$ parametrize sol'n set



© Linear Transformations (LTs)

$T: V \to W$ is a LT if, for all $\vec{v}_1, \vec{v}_2$ in V and any scalar c,

Image of sum = sum of images

Can add, then map or map, then add $\to$ same result

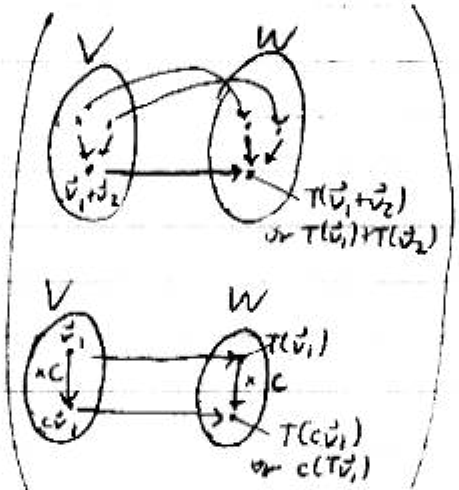
① $T(\vec{v}_1 + \vec{v}_2) = \underbrace{T(\vec{v}_1)} + \underbrace{T(\vec{v}_2)}$

(add in V, 1st then, map to W) (map to W, 1st then, add in W)

You can rescale before or after T is applied $\to$ same result.

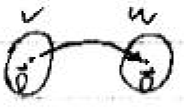
② $T(c\vec{v}_1) = c\,\underbrace{T(\vec{v}_1)}$

(×c in V, 1st then, map to W) (map to W, 1st then, ×c)



Key Props.

$T(\vec{0}_V) = \vec{0}_W$



☆ $T(c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_n\vec{v}_n) = c_1 T(\vec{v}_1) + \ldots + c_n T(\vec{v}_n)$

"$\vec{v}_i$"s are in V

This leads to...

① LTs and Bases

If we know $T(\vec{b}_1), T(\vec{b}_2), \ldots, T(\vec{b}_n)$ for ~~any~~ one basis $\{\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n\}$ of V, then we can find $T(\vec{v})$ for any $\vec{v}$ in V.

If we know how T maps a basis of V, we know how T maps everything in V.

Ex Let $T: \mathbb{R}^2 \to \mathbb{R}^3$ be a LT such that

$$T(1,0) = (1,3,-2)$$
$$T(0,1) = (4,2,0)$$

Find $T(-3,2)$

You don't have to do this

I'll be switching between row vecs, col vecs - it often doesn't matter in this material here.

$$T\left(\begin{bmatrix}-3\\2\end{bmatrix}\right) = T\left(\begin{bmatrix}-3\\0\end{bmatrix} + \begin{bmatrix}0\\2\end{bmatrix}\right)$$

$$= T\left(-3\begin{bmatrix}1\\0\end{bmatrix} + 2\begin{bmatrix}0\\1\end{bmatrix}\right)$$

$$\overset{\star}{=} -3T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) + 2T\left(\begin{bmatrix}0\\1\end{bmatrix}\right)$$

$$= -3(1,3,-2) + 2(4,2,0)$$

$$= (-3,-9,6) + (8,4,0)$$

$$= (5,-5,6)$$

You can find any $T(v_1, v_2)$

(E) The L.T Given by a Matrix

$T(\vec{v}) = \overset{m\times n}{A}\vec{v}$ defines a L.T

$T: \mathbb{R}^n \to \mathbb{R}^m$

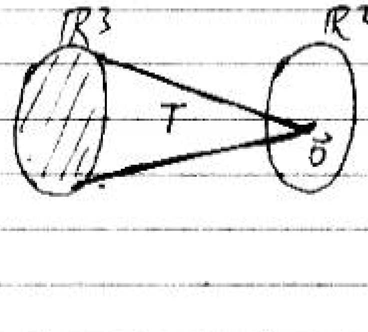
$$ {}_{m}\begin{bmatrix} & A & \end{bmatrix}^{n} \underbrace{\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}}_{\vec{v} \text{ in } \mathbb{R}^n} = \underbrace{\begin{bmatrix} w_1 \\ \vdots \\ w_m \end{bmatrix}}_{T(\vec{v}) \text{ in } \mathbb{R}^m} $$

(column vectors)

<u>Ex</u> If $A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{2\times 3}$, then

$T(\vec{v}) = A\vec{v}$ defines the <u>zero transformation</u>

$T: \mathbb{R}^3 \to \mathbb{R}^2$

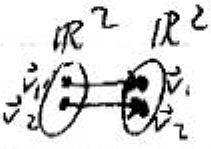


<u>Ex</u> If $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, then

$T(\vec{v}) = A\vec{v}$ defines the <u>identity transformation</u>

$T: \mathbb{R}^2 \to \mathbb{R}^2$

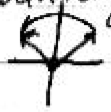
Here, $T(\vec{v}) = \vec{v}$ for all $\vec{v}$ in $V = W = \mathbb{R}^2$



(Do Last)

<u>Ex</u> If $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, then (see Section 12.4, Calc II, Swokowski on rotated conics)

$T: \mathbb{R}^2 \to \mathbb{R}^2$ rotates every vector in $\mathbb{R}^2$ counterclockwise about the origin. angle θ

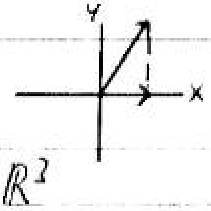


(Optional proof - p.332)

<u>Ex</u> If $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, then

$T: \mathbb{R}^2 \to \mathbb{R}^2$ is a <u>projection</u> in $\mathbb{R}^2$.

$T(\vec{v}) = A\vec{v} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ 0 \end{bmatrix}$



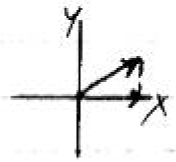
<u>Ex 8</u> - p.332: $\mathbb{R}^3$

Ex Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

let $T(\vec{v}) = A\vec{v}$ define $T: \mathbb{R}^2 \to \mathbb{R}^2$

a) find the image of (4,3).

$$T(4,3) = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 4 \\ 3 \end{bmatrix}}_{\vec{v}} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$$



b) Find the preimage of (3,0)

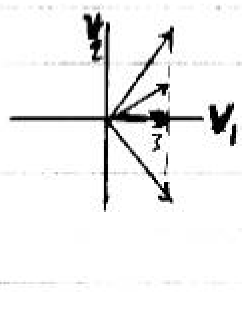
$$T(\vec{v}) = A\vec{v}$$
$$A\vec{v} = T(\vec{v})$$

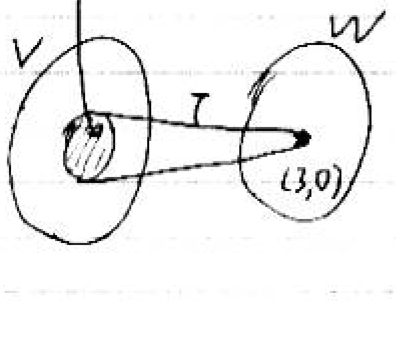
$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

$$\left(\left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 0 & 0 \end{array}\right] \right)$$

$v_1 = 3$
$v_2 = t$ (free)

$\{(3,t) \mid t \text{ is a real } \#\}$



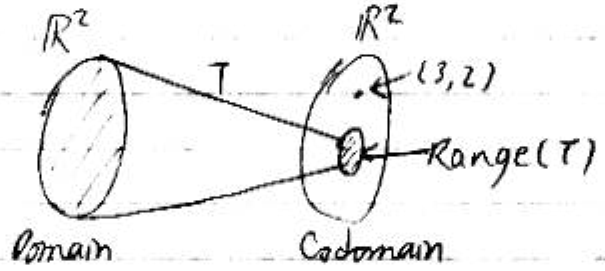


c) Find the *preimage* of (3, 2)

$$\left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 0 & 2 \end{array}\right]$$

Preimage = ϕ

He doesn't get to play. He's left out.



Go back to rotation Ex (6.1.6)