6.2: KERNEL + RANGE

Let $T: V \to W$ be a L.T.

Ⓐ Ker.(T)

= kernel of T

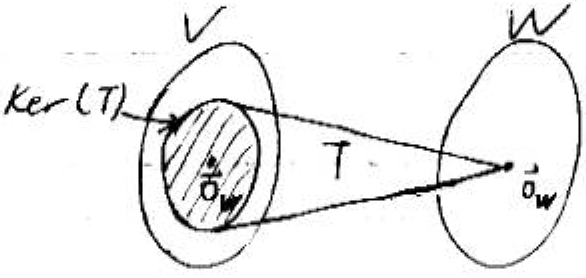
= $\{\vec{v} \text{ in } V \mid T(\vec{v}) = \vec{0}_w\}$

= set of vectors in V that T maps to $\vec{0}$ in W.

is a subspace of V (pf. pp. 339-40)

what vector in V must be in the kernel?

subspace



Ex $T: \mathbb{R}^2 \to \mathbb{R}^2$, find Ker(T) if

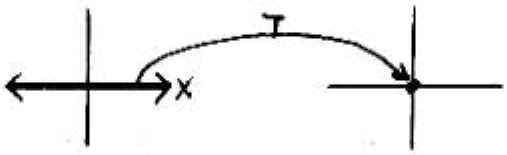
$T(x,y) = \underbrace{(0,y)}_{\text{what makes this} = \vec{0}?}$ ← projection on y-axis

$$\begin{cases} \cancel{0=0} \\ y=0 \end{cases}$$

Your book sticks w/x

$$\begin{cases} x=t \text{ (free)} \\ y=0 \end{cases}$$

$$\boxed{\begin{aligned} Ker(T) &= \{(t,0) \mid t \text{ is a real \#}\} \\ &= x\text{-axis} \end{aligned}}$$



$\underline{T(\vec{x})=\overset{m\times n}{A}\vec{x} \text{ defines } T:\mathbb{R}^n \to \mathbb{R}^m}$ (A LT can be given by a matrix.)

$$\begin{aligned} Ker(T) &= \{\vec{x} \text{ in } \mathbb{R}^n \mid A\vec{x}=\vec{0}\} \\ &= N(A), \text{ the } \underline{\text{nullspace}} \text{ of } A \end{aligned}$$

Its dimension = nullity (T)
or nullity (A)

Ex $T: \mathbb{R}^3 \to \mathbb{R}^2$
$T(\vec{x}) = A\vec{x}$, $\vec{x}$ in $\mathbb{R}^3$

$$A = \begin{bmatrix} 2 & 4 & 6 \\ 0 & 1 & 3 \end{bmatrix}$$

Find a basis for Ker (T) as a subspace of $\mathbb{R}^3$.

Solution

Solve $A\vec{x} = \vec{0}$ and find a basis for $N(A)$.

$$\left[\begin{array}{ccc|c} 2 & 4 & 6 & 0 \\ 0 & 1 & 3 & 0 \end{array}\right]$$

$$\overset{RRE}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & -3 & 0 \\ 0 & 1 & \underset{\substack{x_3 \\ \text{free}}}{3} & 0 \end{array}\right]$$

$$\begin{cases} x_1 \quad -3x_3 = 0 \\ \quad x_2 + 3x_3 = 0 \end{cases}$$

$$\begin{cases} x_1 = 3x_3 \\ x_2 = -3x_3 \end{cases}$$

$$x_3 \stackrel{.}{=} t$$

$$\begin{cases} x_1 = 3t \\ x_2 = -3t \\ x_3 = t \end{cases}$$

$$\left.\vec{x} = t\begin{bmatrix}3\\-3\\1\end{bmatrix}, \ t \text{ is any real \#}\ \right\} \begin{array}{l}\text{Ker}(T):\\ \text{line in } \mathbb{R}^3\end{array}$$

$$\text{Basis}: \boxed{\left\{\begin{bmatrix}3\\-3\\1\end{bmatrix}\right\}}$$

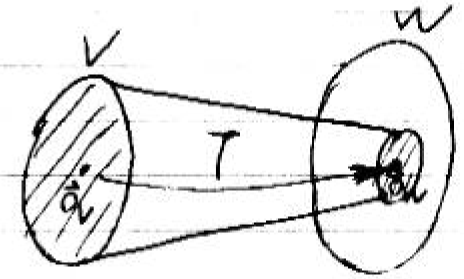
$$\text{nullity}(T) = 1$$

up to 6 Read Exs 1-6

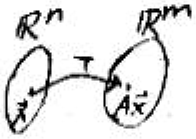
Ⓑ Range(T)

$$= \{T(\vec{v}) \mid \vec{v} \text{ is in } V\} \quad \text{(images)}$$

is a subspace of W



$\underline{T(\vec{x}) = \overset{m\times n}{A}\vec{x}}$



Then, $\text{Range}(T) = \{A\vec{x} \mid \vec{x} \text{ is in } \mathbb{R}^n\}$

$= \{\vec{b} \text{ in } \mathbb{R}^m \mid A\vec{x} = \vec{b} \text{ is consistent}\}$

$= \text{Col}(A)$, the <u>column space</u> of A

Its dimension $=$ rank (T) or rank(A)

<u>Ex</u> $T: \mathbb{R}^4 \to \mathbb{R}^3$
$T(\vec{x}) = A\vec{x}$, $\vec{x}$ in $\mathbb{R}^4$

$$A = \begin{bmatrix} 1 & 2 & 4 & 3 \\ 1 & 2 & 5 & 5 \\ 2 & 4 & 8 & 6 \end{bmatrix}$$

Find a basis for Range(T) as a subspace of $\mathbb{R}^3$.

<u>Solution</u>

$\to$ row-echelon "shape"

$$A \sim \begin{array}{c} \begin{bmatrix} \textcircled{1} & 2 & 4 & 3 \\ 0 & 0 & \textcircled{1} & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{matrix} \uparrow & & \uparrow & \end{matrix} \end{array}$$

Cols 1,3 are pivot cols.

Take cols. 1,3 <u>from A</u>.

$$\text{Basis} = \boxed{\left\{ \underset{\vec{v}_1}{\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}}, \underset{\vec{v}_2}{\begin{bmatrix} 4 \\ 5 \\ 8 \end{bmatrix}} \right\}}$$

Up to 11

Also, $\text{Range}(T) = \text{Span}(\{\vec{v}_1, \vec{v}_2\}) =$ "plane in $\mathbb{R}^3$"
$\text{rank}(T) = 2$

Ⓒ Sum Formulas

(4.6f)

$$\begin{array}{ccccc} \text{rank}(T) & + & \text{nullity}(T) & = & n \\ \updownarrow & & \updownarrow & & \updownarrow \\ \dim(\text{Range}(T)) & + & \dim(\text{Ker}(T)) & = & \dim(\text{domain}, V) \end{array}$$

<u>$T(\vec{x}) = A\vec{x}$</u>

$$[A] \sim \overset{n = \#\text{cols}}{\overset{\downarrow\quad\downarrow\quad\downarrow}{\begin{bmatrix} \text{row-ech} \\ \text{shape} \end{bmatrix}}}.$$

$\text{rank}(T) = \#\text{pivot cols}$
$\text{nullity}(T) = \#\text{nonpivot cols}$

<u>Read Exs. 8, 9</u>

Ex $T: \mathbb{R}^3 \to \mathbb{R}^2$

$T(\vec{x}) = A\vec{x}$

from Ker(T)

$$A \sim \begin{bmatrix} \textcircled{2} & 4 & 6 \\ 0 & \textcircled{1} & 3 \end{bmatrix}$$

$$\left(\sim \begin{bmatrix} \textcircled{1} & 0 & -3 \\ 0 & \textcircled{1} & 3 \end{bmatrix} \right)$$

Be aware of variations of same ?

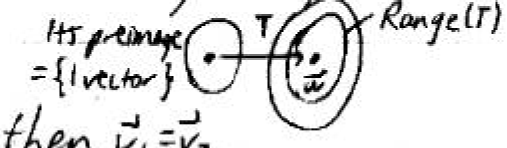
$$\text{rank}(T) = \dim(\text{Range}(T)) = \textcircled{2}$$

$$\text{nullity}(T) = \dim(\text{Ker}(T)) = n - \text{rank}(T) = 3 - 2 = \textcircled{1}$$

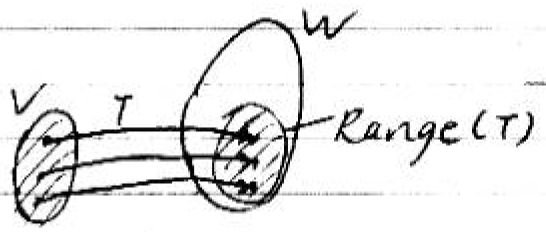
Up to 35

① When is T "One-to-One" (1-1)?

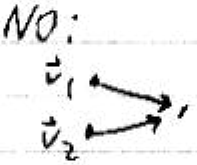
↔ For each $\vec{w}$ in Range (T),



↔ whenever $T(\vec{v}_1) = T(\vec{v}_2)$, then $\vec{v}_1 = \vec{v}_2$



NO:



can't have 2 vectors w/ same image

$\boxed{\text{A LT is 1-1} \Leftrightarrow Ker(T) = \{\vec{0}_W\}}$ Proof (optional) p.344

only $\vec{0}_V \longrightarrow \vec{0}_W$

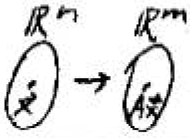
p.344

Read Ex 10

not in book!

$T(\vec{x}) = \overset{m\times n}{A}\vec{x}$

$\mathbb{R}^n \quad \mathbb{R}^m$



"T is 1-1" means "$A\vec{x} = \vec{b}$ has 0 or 1 sol'ns for every $\vec{b}$ in $\mathbb{R}^m$"

↕ p.344

"$Ker(T) = \{\vec{0}\}$" means "$A\vec{x} = \vec{0}$ has only the $\vec{0}$ sol'n" $\underset{\vec{0}_{\mathbb{R}^n}}{\text{only}} \longrightarrow \vec{0}_{\mathbb{R}^m}$

↕

Each $\underset{(LI)}{\text{col.}}$ of a row-ech. shape of A has a PP

Ex $[A] \underset{\text{shape}}{\overset{\text{Row-}}{\underset{\text{ech.}}{\sim}}} \begin{bmatrix} \textcircled{1} & 0 \\ 0 & \textcircled{1} \\ 0 & 0 \end{bmatrix}$

$[A|\vec{0}] \sim \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \begin{array}{l} \Rightarrow x_1 = 0 \\ \Rightarrow x_2 = 0 \\ \\ \end{array}$

$\left.\begin{array}{l} T: \mathbb{R}^2 \to \mathbb{R}^3 \\ T(\vec{x}) = A\vec{x} \end{array}\right\}$ is 1-1 (rank = 2)

$\boxed{\text{A LT is 1-1} \Leftrightarrow \underset{(\#\text{pivot cols.} = \#\text{cols.})}{rank(T) = n}}$

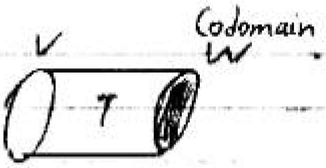
Why?

$rank(T) + \underbrace{nullity(T)}_{\dim(Ker(T))} = n$

$= 0 \Leftrightarrow T$ is 1-1

6.2.9

Ⓔ When is T "Onto"?



↔ each $\vec{w}$ in W has a nonempty preimage → $\dot{\vec{w}}$

Range = Codomain

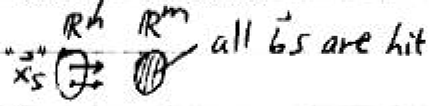
↔ Range(T) = W

↔ $\underset{= \text{dim of Range}}{\text{rank}(T)} = \dim(W)$

If $T(\vec{x}) = A\vec{x}$

T is onto ⇔ $A\vec{x} = \vec{b}$ is consistent for every $\vec{b}$ in $\mathbb{R}^m$

⇔ A's cols. span $\mathbb{R}^m$



not in book

— is what?

⇔ Each row of a row-ech. shape of A has a ⓅⓅ

Ex $T: \mathbb{R}^3 \to \mathbb{R}^{\textcircled{2}}$ dim(W)

$T(\vec{x}) = A\vec{x}$

$$[A] \xrightarrow[\text{shape}]{\text{row-ech.}} \begin{bmatrix} \textcircled{2} & 1 & 4 \\ 0 & \textcircled{3} & 5 \end{bmatrix}$$ each row has a ⓅⓅ

T is onto

Ⓕ What if dim(V) = dim(W)?

→ A linear T: V → W

① is one-to-one
② is onto

is either **both** or **neither**

(≈ basis)

6.2.10

<u>Ex</u> $T : \mathbb{R}^3 \to \mathbb{R}^3$
$T(\vec{x}) = A\vec{x}$

$$[A] \sim \begin{bmatrix} \boxed{2} & -2 & 1 \\ 0 & \boxed{3} & 5 \\ 0 & 0 & \boxed{1} \end{bmatrix}$$

We have a full "set" of $\boxed{PP}$s.
(down the main diagonal)
so T is 1-1 and onto. T is an <u>isomorphism</u>.

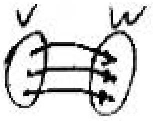
<u>Read Ex 11</u>

, webster
iso - equal, homog., uniform
morph - form
Gr - intellectuals
Romans - engineers, lawyers

<u>⑥ Isomorphisms of Vector Spaces</u>

V and W are <u>isomorphic</u> if there is
a one-to-one and onto linear $T : V \to W$
that "marries" the vectors of V
with the vectors of W.

no bigamy



Me:
You can have
∞-dim V.S. that
are isom., but
that story's more
complicated

$\Leftrightarrow \dim(V) = \dim(W)$ (if $< \infty$)

a basis in your
hot lil hands
could be OG, ON

<u>Ex</u> $\mathbb{R}^3 \cong M_{3,1} \cong M_{1,3} \cong P_2 \cong$ a 3-D subspace of $\mathbb{R}^4$

or $c_3 + c_2 x + c_1 x^2$

"Natural" correspondences $\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \leftrightarrow \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \leftrightarrow [c_1 \; c_2 \; c_3] \leftrightarrow c_1 + c_2 x + c_3 x^2 \leftrightarrow$
$$\left\{ \begin{array}{l} \text{Basis} : \{\vec{b}_1, \vec{b}_2, \vec{b}_3\} \\ T : \text{This space} \to \mathbb{R}^3 \\ T(\underbrace{c_1\vec{b}_1 + c_2\vec{b}_2 + c_3\vec{b}_3}_{\text{a vector in the space}}) = \underset{\uparrow \\ \text{coords in } \mathbb{R}^3}{\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}} \end{array} \right.$$

Adding vectors
in $\mathbb{R}^3$ is
equiv. to
adding polys.
in P_2!

These spaces "act the same" wrt (with respect to) "vector +", "scalar mult."

<u>Read Ex 12</u>