6.3: MATRICES FOR LTs

Ⓐ The Standard Basis for $\mathbb{R}^n$

$$\{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$$

$$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \vec{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

If we know how a linear T maps a <u>basis</u> of $\mathbb{R}^n$, we know how T maps <u>everything</u> in $\mathbb{R}^n$.

Ⓑ The Standard Matrix for $T: \mathbb{R}^n \to \mathbb{R}^m$ (linear)

$$A = \begin{array}{cc} & \begin{array}{cccc} 1 & & & n \end{array} \\ \begin{array}{c} 1 \\ m \end{array} & \begin{bmatrix} T(\vec{e}_1) & T(\vec{e}_2) & \cdots & T(\vec{e}_n) \end{bmatrix} \end{array}$$

Column i is $T(\vec{e}_i)$, the image of $\vec{e}_i$.
(the i^{th} st. basis vector)

Then, $T(\vec{v}) = A\vec{v}$ for <u>every</u> $\vec{v}$ in $\mathbb{R}^n$.

Why?

I'll often switch between row vecs, col vecs - ok here.

$$T(\vec{v}) = T\left(\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}\right)$$
$$= T(v_1\vec{e}_1 + \dots + v_n\vec{e}_n)$$
$$= v_1 T(\vec{e}_1) + \dots + v_n T(\vec{e}_n)$$

$$A\vec{v} = [T(\vec{e}_1) \ \cdots \ T(\vec{e}_n)]\underbrace{\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}}_{\text{weights}}$$
$$\overset{\text{Trust me}}{=} v_1 T(\vec{e}_1) + \dots + v_n T(\vec{e}_n)$$

same

Ⓒ Examples

Ex $T: \mathbb{R}^2 \to \mathbb{R}^3$
$T(x,y) = (x+y,\ 2x,\ 3x-4y)$

Method 1 (Know!)

$$T(\vec{e}_1) = T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$$
$$= (1+0,\ 2(1),\ 3(1)-4(0))$$
$$= (1,\ 2,\ 3)$$

$$T(\vec{e}_2) = T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$$
$$= (0+1,\ 2(0),\ 3(0)-4(1))$$
$$= (1,\ 0,\ -4)$$

$$A = [T(\vec{e}_1)\ T(\vec{e}_2)] = \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ 3 & -4 \end{bmatrix}$$

Method 2 (Shortcut)

Write components of $T(x,y)$ in rows
(line up like terms)

$$\begin{matrix} x & + & y \\ 2x & & \\ 3x & - & 4y \end{matrix}$$

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ 3 & -4 \end{bmatrix}$$

Ex 1 (p.351) Shortcut "Warning"

$T: \mathbb{R}^3 \to \mathbb{R}^2$
$T(x,y,z) = (x-2y, 2x+y)$

$$\begin{matrix} x & - & 2y & + & 0z \\ 2x & + & y & + & 0z \end{matrix}$$ Don't forget z column

$$A = \begin{bmatrix} 1 & -2 & 0 \\ 2 & 1 & 0 \end{bmatrix}$$

projection on x-axis in $\mathbb{R}^2$ Read Ex 2 (p.351-2)

Ex $T: \mathbb{R}^2 \to \mathbb{R}^2$, T stretches vectors by a factor of 2.

Method 1 $T(\vec{e}_1) = T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}2\\0\end{bmatrix}$

$T(\vec{e}_2) = T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}0\\2\end{bmatrix}$

$A = [T(\vec{e}_1) \; T(\vec{e}_2)] = \begin{bmatrix}2 & 0\\0 & 2\end{bmatrix}$

e.g., $\begin{bmatrix}2 & 0\\0 & 2\end{bmatrix}\begin{bmatrix}1\\3\end{bmatrix} = \begin{bmatrix}2\\6\end{bmatrix}$

Method 2 $T(x,y) = (2x, 2y)$
etc.

up to 19

① Extensions

Composition: (Motivates the way we multiply matrices!)

Meyer 3.4
This is why we mult. matrices the way we do.
1855 - Arthur Cayley made this connection w/ comp. of linear funcs.
1857 - A Memoir on the Theory of Matrices - birth of modern linear alg.
(We already knew about dets, Gaussian elim.)

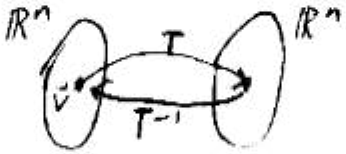
$$\underbrace{\underset{\substack{\text{apply}\\ \text{2nd}\\ \mathbb{R}^m \to \mathbb{R}^p}}{T_2} \circ \underset{\substack{\text{apply}\\ \text{1st}\\ \mathbb{R}^n \to \mathbb{R}^m}}{T_1}}_{\mathbb{R}^n \to \mathbb{R}^p} (\vec{v}) = T_2 \overset{\text{linear}}{(} T_1(\vec{v})) = \underbrace{\underset{p\times m}{A_2}\, \underset{m \times n}{A_1}}_{\substack{\text{standard matrices}\\ p \times n}} \vec{v}$$

Read Ex 3 (p. 353)

Inverse:

$$T^{-1}(\vec{v}) = A^{-1}\vec{v}$$

↑ undoes T



as opposed to nonlinear? ex

Linear T is invertible ⟺ A is invertible
↕
T is one-to-one (1-1) and onto (an isomorphism)

Read Ex 4 (pp. 354-5)