

6.4: TRANSITION MATRICES and SIMILARITY(A) Similar Matrices

$$A, A', A'' - n \times n$$

A' is similar to A ($A' \sim A$) \Leftrightarrow

There exists an invertible $n \times n$ matrix P such that $A' = P^{-1}AP$

Props. Similarity is an equivalence relation. Math 245!

① (A is similar to itself.) (Reflexivity)

$$A \sim A$$

Proof $A = I^{-1}AI$ (Let $P=I$)

② (If A' is similar to A , then A is similar to A' .) (Symmetry)

$$\text{If } A' \sim A \text{ then } A \sim A'$$

Proof $A' \sim A$
 \rightarrow there exists $P: A' = P^{-1}AP$
 $PA'P^{-1} = \underbrace{PP^{-1}}_I \underbrace{APP^{-1}}_I$
 $PA'P^{-1} = A$

$$A = PA'P^{-1}$$

$$A = Q^{-1}A'Q \quad (\text{Let } Q = P^{-1})$$

③ (If A' is similar to A , and A'' is similar to A' , then A'' is similar to A .) (Transitivity)

$$\text{If } A' \sim A \text{ and } A'' \sim A' \text{ then } A'' \sim A$$

Proof #19

A few took 245

Book uses B, C I need for basis

What could I use for P ?

Don't confuse w/ A'

$M_{1,1}$ classes
 $\{C\} | C \in \mathbb{R}$

$M_{n,n}$ can be partitioned into similarity classes.



$M_{n,n}$
 $\begin{bmatrix} p & 1 & -2 & 2 \\ & & & \end{bmatrix}$

(B) Transition Matrices

I figured you had plenty to do in Ch. 4 anyway

(From 4.7)

Let $B' = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ be a basis for \mathbb{R}^n .

Let $B = \{\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n\}$ be the standard basis.

The transition matrix from B' to B is

$$P = [\vec{v}_1 \ \vec{v}_2 \ \dots \ \vec{v}_n]$$

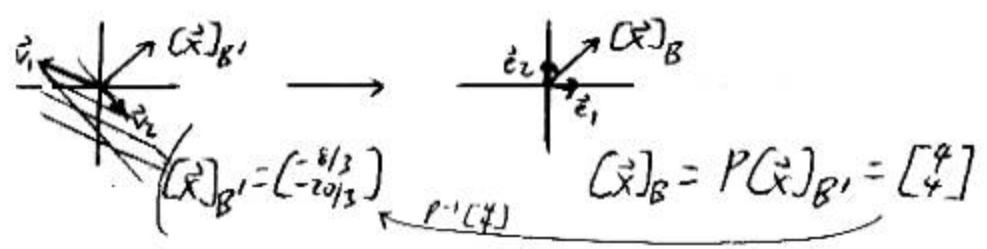
Idea If $[\vec{x}]_{B'}$ are the coords of \vec{x} relative to B' and $[\vec{x}]_B$ then

$$P \underset{B' \rightarrow B}{[\vec{x}]_{B'}} = [\vec{x}]_B$$

$$\rightarrow [\vec{x}]_{B'} = \underset{\substack{\text{transition} \\ \text{matrix} \\ B \rightarrow B'}}{P^{-1}} [\vec{x}]_B$$

Ex $B' = \{(-4, 1), (1, -1)\}$
 $B = \{(1, 0), (0, 1)\}$

$$P = \begin{bmatrix} -4 & 1 \\ 1 & -1 \end{bmatrix}$$



© The Matrix for T Relative to a Nonstandard Basis

Ex (#2)

a) Find the matrix A' for $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$
 $T(x,y) = (x+y, 4y)$
 relative to the basis $B' = \{(-4, 1), (1, -1)\}$.

same B'
as before.

Solution

The standard matrix for T
 (relative to the standard basis) is:

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 4 \end{bmatrix}$$

The transition matrix from B' to B is:

$$P = \begin{bmatrix} -4 & 1 \\ 1 & -1 \end{bmatrix}$$

Then, the transition matrix from B to B' is:

$$P^{-1} = \frac{1}{\det(P)} \begin{bmatrix} -1 & -1 \\ -1 & -4 \end{bmatrix}$$

$$= -\frac{1}{3} \begin{bmatrix} 1 & 1 \\ 1 & 4 \end{bmatrix}$$

$$A' = P^{-1}AP$$

↑
 T
 rel. to B' → B' ③
 ↑ ↑ ↑
 ② T ① B' → B
 rel. to B

A' operates entirely within the context of the new, nonstandard(?) coord. system.

Matrix mult. is assoc.

$$A' = -\frac{1}{3} \begin{bmatrix} 1 & 1 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} -4 & 1 \\ 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{3} & \frac{4}{3} \\ -\frac{13}{3} & \frac{16}{3} \end{bmatrix}$$

Describes T relative to B' .

rel. to this coord system.



± Meyer 256
"maybe" because $A \neq A'$

Two matrices represent the same linear $T: V \rightarrow V$, maybe w/ respect to different bases \longleftrightarrow similar.



Anton 8 ed p.406
Sim. Invariants
Rk, Rank, Nullity,
Trace, Eigen, Char poly, Eigen space dim (not evcs)

What do these similar matrices have in common?

Similarity invariants

trace ($\sum a_{ii}$)

det

rank, nullity

eigenvalues (Ch. 7)

in HW

what's arguably
most important
associated w/
an $n \times n$ matrix?

Meyer 256

quite
captivating
finite base, rank
then some #s
→ conjecture
→ pf (by later!)

Theory goal: Isolate props. of LT and their matrices
that are coord- (basis-) independent.

*We share a philosophy about linear algebra: we think basis-free,
but when the chips are down we close the office door
and compute with matrices like fury.
Irving Kaplansky (1917) speaking about Paul Halmos (1916)*