CH.7: EIGENVALUES and EIGENVECTORS

7.1: INTRO

(A) Definitions

$A - n \times n$

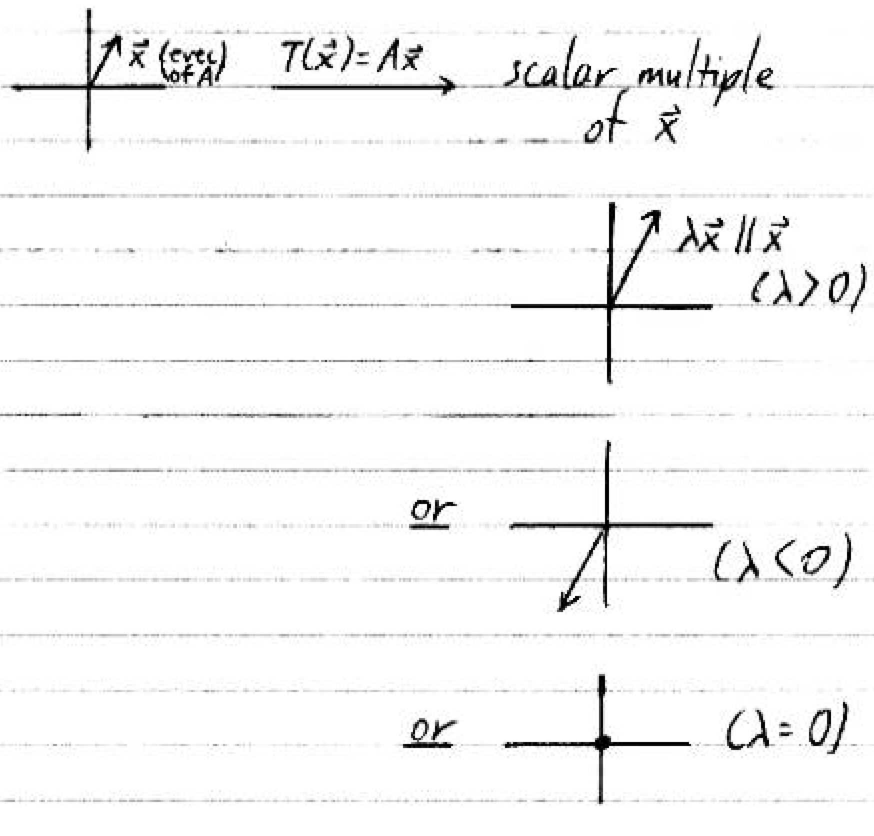
The scalar λ ("lambda") is an <u>eigenvalue</u> (eval) of A $\Leftrightarrow$ there exists $\vec{x} \neq \vec{0}$ such that $A\vec{x} = \lambda\vec{x}$.

$\vec{x}$ is then an <u>eigenvector</u> (evec) of A corresp. to λ.

<u>Idea</u>

If we apply

shrinks if $\lambda = \frac{1}{2}$



(B) Verifying Evals, Evecs

6.1.4 · 20f **Ex** Verify that $\lambda = 3$ is an eval of $A = \begin{bmatrix} 5 & -3 \\ -4 & 9 \end{bmatrix}$ and that $\vec{x} = (3, 2)$ is a corresp. evec.

Solution

$$A\vec{x} = \begin{bmatrix} 5 & -3 \\ -4 & 9 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 9 \\ 6 \end{bmatrix}$$

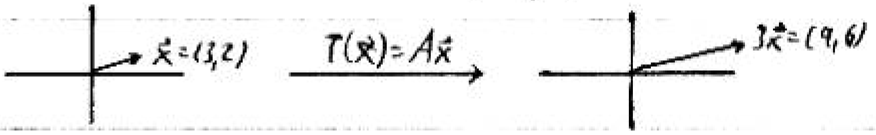
$$\lambda\vec{x} = 3\begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

same

$$= \begin{bmatrix} 9 \\ 6 \end{bmatrix}$$

So, $A\vec{x} = \lambda\vec{x}$ where $\underbrace{\lambda = 3, \vec{x} = (3, 2)}_{\text{eval-evec pair}}$. ✓

3 is a stretching factor.



6.1.6-20F

<u>Ex</u> Is $\vec{x} = (1,-2,1)$ an evec of
$A = \begin{bmatrix} 3 & 6 & 7 \\ 3 & 3 & 7 \\ 5 & 6 & 5 \end{bmatrix}$? If so, find its
corresp. eval.

$$A\vec{x} = \begin{bmatrix} 3 & 6 & 7 \\ 3 & 3 & 7 \\ 5 & 6 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -2 \\ 4 \\ -2 \end{bmatrix}$$

$\left(\text{Is this a scalar multiple of } \vec{x} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}\text{? YES} \right)$

eval-evec pair

$$= \left(-2\vec{x} \text{ or}\right) -2 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

$\vec{x}$ <u>is</u> an evec

its corresp. eval is -2

(C) Eigenspaces

If A $(n\times n)$ has an eval λ, then

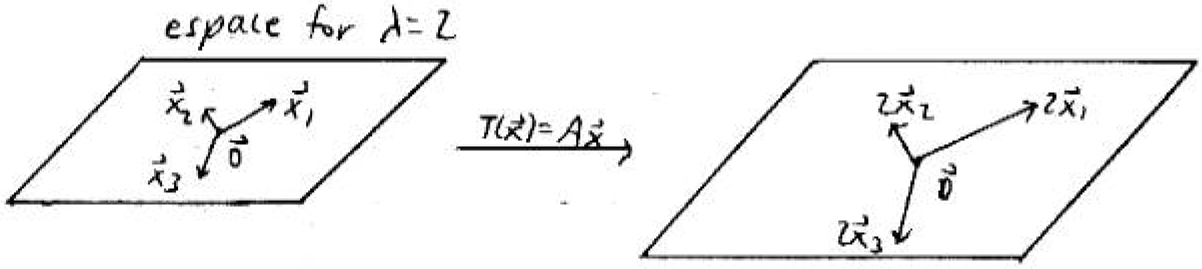
$$E_\lambda = \{\text{all evecs of } \lambda\} \underbrace{\cup \{\vec{0}\}}_{\text{throw in } \vec{0}}$$

is the eigenspace (espace) of λ.

a subspace of $\mathbb{R}^n$

Ex

What happens to $\vec{x}_1$ under this transf?



Proof: E_λ is a subspace of $\mathbb{R}^n$

$E_\lambda \neq \emptyset$, $E_\lambda \subseteq \mathbb{R}^n$ (has $\vec{0}$)

Let $\vec{x}_1, \vec{x}_2$ be in E_λ. (They're evecs of λ or $\vec{0}$.) $A\vec{x}_1 = \lambda\vec{x}_1$, $A\vec{x}_2 = \lambda\vec{x}_2$

Show $\vec{x}_1 + \vec{x}_2$ is in E_λ: (i.e., $A(\vec{x}_1 + \vec{x}_2) = \lambda(\vec{x}_1 + \vec{x}_2)$)

$$A(\vec{x}_1 + \vec{x}_2) = A\vec{x}_1 + A\vec{x}_2 = \lambda\vec{x}_1 + \lambda\vec{x}_2 = \lambda(\vec{x}_1 + \vec{x}_2)$$

$A\vec{x}_1 = \lambda\vec{x}_1$ even if $\vec{x}_1 = \vec{0}$

Show $c\vec{x}_1$ is in E_λ: (c is any scalar) (i.e., $A(c\vec{x}_1) = \lambda(c\vec{x}_1)$)

$$A(c\vec{x}_1) = c(A\vec{x}_1) = c(\lambda\vec{x}_1) = \lambda(c\vec{x}_1)$$

① Finding Evals, Evecs

A - $n \times n$

When does $A\vec{x} = \lambda\vec{x}$ have nontrivial sol'ns $\vec{x}$ (evecs)?

$$
\begin{aligned}
& A\vec{x} = \lambda\vec{x} \\
\Leftrightarrow\ & A\vec{x} = \lambda(I\vec{x}) \\
\Leftrightarrow\ & \vec{0} = \lambda I\vec{x} - A\vec{x} \\
\Leftrightarrow\ & \vec{0} = (\lambda I - A)\vec{x}
\end{aligned}
$$

More precisely, $I = I_n$.

or: $A\vec{x} - \lambda I\vec{x} = \vec{0}$

$(A - \lambda I)\vec{x} = \vec{0}$

When does $\underbrace{(\lambda I - A)}_{n \times n}\vec{x} = \vec{0}$ have nontrivial sol'ns $\vec{x}$?

When det = ?

IMT

When $\underbrace{\det(\lambda I - A) = 0}_{\text{i.e., singular/noninvertible}}$ (Some books: $\det(A - \lambda I) = 0$)

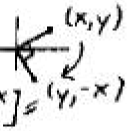
<u>How do we find evals?</u>

Solve $\underbrace{\underbrace{\det(\lambda I - A)}_{\text{characteristic polynomial of } A} = 0}_{\text{characteristic equation of } A}$ for λ.

Meyer 549: If A real skew-sym ($A^T = -A$) $\Rightarrow$ A has pure imag. evals

Maybe no real evals!

<u>Ex</u> $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$



No evecs. for real evals.

$A\begin{bmatrix} x \\ y \end{bmatrix} = (y, -x)$

$\det(\lambda I - A) = |\lambda I - A| = \begin{vmatrix} \lambda & -1 \\ 1 & \lambda \end{vmatrix}$

$= \lambda^2 + 1$ has no real zeros/roots. No real evals!

<u>How do we find the evecs corresp. to λ?</u>

We take the nonzero sol'ns of

$$(\lambda I - A)\vec{x} = \vec{0}$$

<u>Ex</u> Find the evals and corresp. evecs of $A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$.

Do <u>not</u> apply EROs now! EROs do not necessarily preserve evals.

<u>Find evals</u>

Solve $|\lambda I - A| = 0$

$$\left| \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} \right| = 0$$

Here, in principle, you can apply EROs - we're talking about dets.

$$\begin{vmatrix} \lambda - 1 & -2 \\ -3 & \lambda - 2 \end{vmatrix} = 0$$

Sometimes (n odd?) leading coeff. is +1, not −1 if do $|\lambda I - A|$ vs. $|A - \lambda I|$

$$\underbrace{\begin{aligned} (\lambda - 1)(\lambda - 2) - 6 &= 0 \\ \lambda^2 - 3\lambda + 2 - 6 &= 0 \\ \underbrace{\lambda^2 - 3\lambda - 4}_{\text{char. poly. of } A} &= 0 \end{aligned}}_{\text{char. eq. of } A}$$

$$(\lambda-4)(\lambda+1)=0$$ ← or QF (Quadratic formula)

$$\lambda_1 = 4, \quad \lambda_2 = -1$$

Find the evecs for $\lambda_1 = 4$

Find the nonzero sol'ns of

$$(\lambda I - A)\vec{x} = \vec{0}$$

$$\left[\begin{array}{c|c} 4I - A & \vec{0} \end{array}\right]$$

write $4I$ 1st

$$\left[\begin{array}{cc|c} 4-1 & 0-2 & 0 \\ 0-3 & 4-2 & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 3 & -2 & 0 \\ -3 & 2 & 0 \end{array}\right]$$

You better have a row of 0s!! [square $|\vec{0}$]

$$\sim \left[\begin{array}{cc|c} 3 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$ ← You better have a row of "0"s!

$$\overset{RRE}{\sim} \left[\begin{array}{cc|c} 1 & -\frac{2}{3} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\begin{cases} x_1 - \frac{2}{3}x_2 = 0 \\ \cancel{0 = 0} \end{cases}$$

$$\left\{ x_1 = \frac{2}{3}x_2 \right.$$

Let $x_2 = t$

$$\begin{cases} x_1 = \frac{2}{3}t \\ x_2 = t \end{cases}$$

$$\vec{x} = t\begin{bmatrix} 2/3 \\ 1 \end{bmatrix}, \quad t \neq 0$$

Shortcut when 1 free var:
$\frac{2}{3}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 1\begin{bmatrix} -2/3 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$
soln $\leftrightarrow$ cols from RRE form
as weights, we get RHS

($\vec{0}$ isn't an evec)

<u>Find the evecs for $\lambda_2 = -1$</u>

$$\left[(-1)I - A \,\middle|\, \vec{0} \right]$$

$$\left[\begin{array}{cc|c} -1-1 & 0-2 & 0 \\ 0-3 & -1-2 & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} -2 & -2 & 0 \\ -3 & -3 & 0 \end{array}\right]$$

$$\overset{RRE}{\sim} \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$x_1 + x_2 = 0$$

$$\Rightarrow x_1 = -x_2$$

Let $x_2 = t$

$$\begin{cases} x_1 = -t \\ x_2 = t \end{cases}$$

$$\vec{x} = t\begin{bmatrix}-1\\1\end{bmatrix},\ t \neq 0$$

Shortcut: $(-1)\begin{bmatrix}1\\0\end{bmatrix} + (1)\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$

In general, the char. poly. is n^{th}-degree in λ. A can have at most n real distinct evals.

If $n \geq 3$, I will give you the evals, unless...
see my Handout

Ⓔ Triangular Matrices

Their evals are the main diagonal entries.

Ex
$$A = \begin{bmatrix} 1 & 0 & 0 \\ 4 & -2 & 0 \\ 5 & 9 & 0 \end{bmatrix}$$

is lower triangular.

Evals: $1, -2, 0$

Why?

$$A = \begin{bmatrix} d_1 & & 0 \\ & d_2 & \\ & & d_n \end{bmatrix}$$

$$|\lambda I - A| = \begin{vmatrix} \lambda - d_1 & & 0 \\ & \lambda - d_2 & \\ & & \lambda - d_n \end{vmatrix}$$

$$= (\lambda - d_1)(\lambda - d_2)\cdots(\lambda - d_n) \stackrel{\text{set}}{=} 0$$

$$\lambda = d_1,\ \lambda = d_2, \ldots, \lambda = d_n$$

Similarly for ◸

(F) More Exs.

8.1.20 Ex $\lambda = 7$ is one of the evals of

$$A = \begin{bmatrix} 7 & -4 & 4 \\ -4 & 5 & 0 \\ 4 & 0 & 9 \end{bmatrix}$$

Find the evecs corresp. to $\lambda = 7$.

$$\left[\underset{7}{\lambda} I - A \,\middle|\, \begin{matrix} 0 \\ 0 \\ 0 \end{matrix}\right]$$

$$\left[\begin{matrix} 7-7 & 0-(-4) & 0-4 \\ 0-(-4) & 7-5 & 0-0 \\ 0-4 & 0-0 & 7-9 \end{matrix} \,\middle|\, \begin{matrix} 0 \\ 0 \\ 0 \end{matrix}\right]$$

$$\left[\begin{matrix} 0 & 4 & -4 \\ 4 & 2 & 0 \\ -4 & 0 & -2 \end{matrix} \,\middle|\, \begin{matrix} 0 \\ 0 \\ 0 \end{matrix}\right]$$

$$\underset{\sim}{\text{RRE}} \left[\begin{matrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{matrix} \,\middle|\, \begin{matrix} 0 \\ 0 \\ 0 \end{matrix}\right]$$

$\uparrow$ x_3 free

$$\begin{cases} x_1 \quad + \frac{1}{2}x_3 = 0 \\ \quad x_2 - x_3 = 0 \end{cases}$$

$$\begin{cases} x_1 = -\frac{1}{2}x_3 \\ x_2 = x_3 \end{cases}$$

Let $x_3 = t$

$$\begin{cases} x_1 = -\frac{1}{2}t \\ x_2 = t \\ x_3 = t \end{cases}$$

$-\frac{1}{2}\begin{bmatrix}1\\0\\0\end{bmatrix}+$

$$\vec{x} = t\begin{bmatrix} -1/2 \\ 1 \\ 1 \end{bmatrix}, t \neq 0$$ Shortcut

(Espaces don't have to be 1-dim!!)

<u>Ex 6 (p. 386)</u>

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 5 & -10 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 3 \end{bmatrix}$$

<u>Find evals</u> (I'd give them)

Solve $|\lambda I - A| = 0$

$(\lambda-1)^{2}(\lambda-2)(\lambda-3)=0$

$\lambda_1 = 1$ has (algebraic) multiplicity 2
$\lambda_2 = 2$ " " 1
$\lambda_3 = 3$ " " 1

The espace for $\lambda_1 = 1$ could be 1-dim or 2-dim

When you solve

$$\left[(1)I - A \mid \vec{0}\right]$$

$$\vec{x} = s\begin{bmatrix}0\\1\\0\\0\end{bmatrix} + t\begin{bmatrix}-2\\0\\2\\1\end{bmatrix}$$

The espace for $\lambda_1 = 1$ is 2-dim.

form a basis (for the espace for $\lambda_1 = 1$)

Ex $\begin{bmatrix}1 & & & \\ & 4 & & \\ & & 4 & \\ 0 & & & 4\end{bmatrix}$

$\lambda = 4$ has (alg) multiplicity 3
Its espace can have dim. 1, 2, or 3.
geometric mult'y.

Anton 353
3 = algebraic mult.
dim(espace) = geometric mult.
Isn't it confusing?
What could dim of espace be?

MATH 254: NOTES ON 7.1

How do we find eigenvalues for large matrices?

If a matrix is upper or lower triangular, its eigenvalues are simply the entries along the main diagonal.

In Example 6 on pp.386-7, we get relatively lucky with the matrix A. Cofactor expansions can be used to expand $|\lambda I - A|$. If you exploit "0"s along the way, the expansion is quick and easy. It turns out that $|\lambda I - A|$ is simply the product of the diagonal entries of $\lambda I - A$. Of course, we're not always so lucky!

7.1, #21

We will find the eigenvalues of $A = \begin{bmatrix} 0 & -3 & 5 \\ -4 & 4 & -10 \\ 0 & 0 & 4 \end{bmatrix}$.

This problem is similar to Example 8 on p.389. We luck out in that the third row has a couple of "0"s, so we can use it as our "magic row" in our cofactor expansion.

$$\begin{aligned}
|\lambda I - A| &= \begin{vmatrix} \lambda & 3 & -5 \\ 4 & \lambda - 4 & 10 \\ 0 & 0 & \lambda - 4 \end{vmatrix} \\
&= +(\lambda - 4)\begin{vmatrix} \lambda & 3 \\ 4 & \lambda - 4 \end{vmatrix} \\
&= (\lambda - 4)\left[\lambda(\lambda - 4) - 12\right] \\
&= (\lambda - 4)\left(\lambda^2 - 4\lambda - 12\right) \\
&= \underbrace{(\lambda - 4)(\lambda - 6)(\lambda + 2)}_{\text{characteristic polynomial}}
\end{aligned}$$

The eigenvalues of A are the roots of its characteristic polynomial: 4, 6, and -2.

SEE BACK

7.1, #19

We will find the eigenvalues of $A = \begin{bmatrix} 1 & 2 & -2 \\ -2 & 5 & -2 \\ -6 & 6 & -3 \end{bmatrix}$.

$$|\lambda I - A| = \begin{vmatrix} \lambda - 1 & -2 & 2 \\ 2 & \lambda - 5 & 2 \\ 6 & -6 & \lambda + 3 \end{vmatrix}$$

We can expand along the first row.

$$= +(\lambda - 1)\begin{vmatrix} \lambda - 5 & 2 \\ -6 & \lambda + 3 \end{vmatrix} \qquad -(-2)\begin{vmatrix} 2 & 2 \\ 6 & \lambda + 3 \end{vmatrix} \qquad +(2)\begin{vmatrix} 2 & \lambda - 5 \\ 6 & -6 \end{vmatrix}$$

$$= (\lambda - 1)\left[(\lambda - 5)(\lambda + 3) - (-12)\right] + 2\left[(2)(\lambda + 3) - 12\right] + 2\left[-12 - (6)(\lambda - 5)\right]$$

$$= (\lambda - 1)\left[\lambda^2 - 2\lambda - 15 + 12\right] \qquad + 2[2\lambda + 6 - 12] \qquad + 2[-12 - 6\lambda + 30]$$

$$= (\lambda - 1)\left[\lambda^2 - 2\lambda - 3\right] \qquad + 2[2\lambda - 6] \qquad + 2[18 - 6\lambda]$$

Expanding this mess out and combining like terms, we get....

$$= \underbrace{\lambda^3 - 3\lambda^2 - 9\lambda + 27}_{\text{characteristic polynomial}}$$

SHORT WAY

We get lucky with this polynomial, believe it or not!
Factoring by grouping works nicely here….

$$\lambda^3 - 3\lambda^2 - 9\lambda + 27 = \left(\lambda^3 - 3\lambda^2\right) + (-9\lambda + 27)$$

$$= \lambda^2(\lambda - 3) - 9(\lambda - 3)$$

We can now factor out $(\lambda - 3)$.

$$= \underbrace{\left(\lambda^2 - 9\right)}_{\text{Factor}}(\lambda - 3)$$

$$= (\lambda + 3)(\lambda - 3)(\lambda - 3)$$

$$= (\lambda + 3)(\lambda - 3)^2$$

Therefore, -3 is an eigenvalue of multiplicity 1, and 3 is an eigenvalue of multiplicity 2 (which makes a two-dimensional eigenspace possible).

LONG WAY (but more general)

Rational Zero Test, or Rational Roots Theorem

Hopefully, you saw this in Math 141 (Precalculus).

If a polynomial $a_n\lambda^n + a_{n-1}\lambda^{n-1} + ... + a_1\lambda + a_0$ (where the "a_i"s are real coefficients, $a_n \neq 0$, and $a_0 \neq 0$) has rational roots, those roots can be obtained from the form $\pm\dfrac{p}{q}$, where p is a factor of a_0, and q is a factor of a_n.

Characteristic polynomials are monic (i.e., their leading coefficient, a_n, is always 1), so their rational roots can be obtained from $\pm p$, where p is a factor of a_0, the constant term.

In our example, the characteristic polynomial is $\lambda^3 - 3\lambda^2 - 9\lambda + 27$.
Therefore, any rational roots of this polynomial (i.e., any rational solutions to the characteristic equation $\lambda^3 - 3\lambda^2 - 9\lambda + 27 = 0$) must be in the following list of factors of 27:

$$\pm 1,\ \pm 3,\ \pm 9,\ \pm 27$$

By trial-and-error, it turns out that 3 is a root of the polynomial (plug it in and see!) Therefore, $(\lambda - 3)$ is a factor of the polynomial. We can use long or synthetic division to find the other factor.

Synthetic Division

Root = 3	1	-3	-9	27	← List the coefficients here
	1				← Bring down the "1"

Root = 3	1	-3	-9	27	
		3			← Multiply the "**1**" by the **Root, 3**
	1	**0**			← Add down the column

Root = 3	1	-3	-9	27	
		3	0		← Multiply the "**0**" by the **Root, 3**
	1	**0**	**-9**		← Add down the column

Root = 3	1	-3	-9	27	
		3	0	-27	← Multiply the "**-9**" by the **Root, 3**
	1	**0**	**-9**	0	← Add down the column

The last "0" that we get is our remainder, so there is a clean factorization.
The boldfaced numbers in the bottom row are the coefficients for the quadratic factor we are looking for: $\lambda^2 + 0\lambda - 9$, or simply $\lambda^2 - 9$.

$$\lambda^3 - 3\lambda^2 - 9\lambda + 27 = (\lambda - 3)\underbrace{(\lambda^2 - 9)}_{\text{Factor}}$$

The Quadratic Formula could be used for "worse" quadratics.

$$= (\lambda - 3)(\lambda + 3)(\lambda - 3)$$
$$= (\lambda - 3)^2(\lambda + 3)$$

Again, 3 is an eigenvalue of multiplicity 2, and -3 is an eigenvalue of multiplicity 1.

7.2: DIAGONALIZATION

A - $n \times n$

(A) Definition

A is <u>diagonalizable</u> (diag'e) $\leftrightarrow$
<u>A is similar to a diagonal matrix</u>

i.e., there exists an invertible $n \times n$ matrix P such that $P^{-1}AP$ is diagonal.

a matrix is similar to itself

<u>Ex</u> If A is diagonal, $I^{-1}AI = A$ is diagonal, so A is diag'e.

<u>Ex</u> Verify that A is diag'e by showing that $P^{-1}AP$ is diagonal.

p. 450 #1

$$A = \begin{bmatrix} 2 & 1 \\ 5 & -2 \end{bmatrix} \quad P = \begin{bmatrix} 1 & 1 \\ -5 & 1 \end{bmatrix}$$

matrix mult. is assoc.

$$P^{-1}AP = \underbrace{\begin{bmatrix} 1 & 1 \\ -5 & 1 \end{bmatrix}^{-1}}_{\frac{1}{\det(P)}\begin{bmatrix} 1 & -1 \\ 5 & 1 \end{bmatrix}} \underbrace{\begin{bmatrix} 2 & 1 \\ 5 & -2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -5 & 1 \end{bmatrix}}_{\begin{bmatrix} -3 & 3 \\ 15 & 3 \end{bmatrix}}$$

flip signs

switch

$$= \frac{1}{6}\begin{bmatrix} 1 & -1 \\ 5 & 1 \end{bmatrix}\begin{bmatrix} -3 & 3 \\ 15 & 3 \end{bmatrix}$$

$$= \frac{1}{6}\begin{bmatrix} -18 & 0 \\ 0 & 18 \end{bmatrix}$$

$$= \begin{bmatrix} -3 & 0 \\ 0 & 3 \end{bmatrix}$$ diagonal ✓
So, A is diag'e.

evals are invariant under sim.

Pf p.393

$|\lambda I - P^{-1}AP|$

$= |P^{-1}\lambda I P - P^{-1}AP|$

$= |P^{-1}(\lambda I - A)P|$

$= |P^{-1}||\lambda I - A||P|$

evals are...what?

ⓑ <u>Similar matrices have the same evals.</u>

If $D = P^{-1}AP$ ← A and D are similar

D diagonal: its evals are on its main diagonal

A has the same evals

<u>Ex</u> In ⓐ

$\left(A = \begin{bmatrix} 2 & 1 \\ 5 & -2 \end{bmatrix} \text{ is similar to} \right)$

$$A \sim D = \begin{bmatrix} -3 & 0 \\ 0 & 3 \end{bmatrix}$$

-3 and 3 are the evals of D <u>and</u> of A

© When is A diag'e?

if you have n LI vecs in an n-dim space, they also span

$\Leftrightarrow$ there exist n LI evecs of A
(i.e., an evec-basis for $\mathbb{R}^n$)

Ⓓ How do you diagonalize A?

We need P, diagonal D such that

$$D = P^{-1}AP$$

① Find n LI evecs of A:

$\vec{p}_1$ (w/ corresp. eval λ_1)
$\vec{p}_2$ λ_2
$\vdots$
$\vec{p}_n$ λ_n ← maybe duplicates

If you <u>can't</u>, then A is not diag'e.
If you <u>can</u> ...

② Let $P = [\vec{p}_1 \ \vec{p}_2 \ \cdots \ \vec{p}_n]$

cols. are the n LI evecs

③ Let $D = P^{-1}AP$ (D, A similar → same evals)

Then, $$D = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & O \\ & & \ddots & \\ O & & & \lambda_n \end{bmatrix}$$

order corresponds to order of evecs in P

Then, we'll do some examples.

Ⓔ LI Theorems

"Different espaces are LI"

Evecs corresponding to different evals (espaces) form a LI set.

If different $\lambda_1 \to \vec{p}_1, \ldots, \lambda_k \to \vec{p}_k$ } then, LI

Each eval has its own espace, with only $\vec{0}$ in common.

Each eval has a 1-dim. espace.

If A has n different real evals, then A is <u>guaranteed</u> to be diag'e, since you can find n LI evecs.

Ⓕ Examples

Cum. Test p. 435, #11

Ex Diagonalize $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 3 & 1 \\ 0 & -3 & -1 \end{bmatrix}$

Evals are: 0, 1, 2 (given).

A is 3×3. A has $n=3$ different evals, so we know A is diag'e.

① Find $n=3$ LI evecs of A

$\lambda_1 = 0$

What are its evecs?

$$\left[\cancel{0I} - A \,\middle|\, \vec{0}\right]$$

$\vdots$

$$\vec{x} = t\begin{bmatrix} -1 \\ -1 \\ 3 \end{bmatrix}, \ t \neq 0$$

Espace for 0 is 1-dim.

Can grab any vector from this 1-D espace

Let $\vec{p}_1 = \begin{bmatrix} -1 \\ -1 \\ 3 \end{bmatrix}$ $\left(\text{or } \begin{bmatrix} -2 \\ -2 \\ 6 \end{bmatrix}, \text{etc.}\right)$

$\underline{\lambda_2 = 1}$

$$\left[\, 1I - A \,\middle|\, \vec{0} \,\right]$$

$$\vdots$$

$$\vec{x} = t \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{\vec{p}_2}, \quad t \neq 0$$

$\underline{\lambda_3 = 2}$

$$\left[\, 2I - A \,\middle|\, \vec{0} \,\right]$$

$$\vdots$$

$$\vec{x} = t \underbrace{\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}}_{\vec{p}_3}, \quad t \neq 0$$

② Let $P = [\vec{p}_1 \ \ \vec{p}_2 \ \ \vec{p}_3]$

$$= \begin{bmatrix} -1 & 1 & 1 \\ -1 & 0 & 1 \\ 3 & 0 & -1 \end{bmatrix}$$

③ Let $D = P^{-1}AP$

Don't even have to find P^{-1} in HW, you're asked to verify

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

If you reorder $\vec{p}_i$'s you must reorder λ_i's

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Bottom line: Give P, D

Lin 6.3.6

Ex Diagonalize $A = \begin{bmatrix} 4 & 0 & -2 \\ 2 & 5 & 4 \\ 0 & 0 & 5 \end{bmatrix}$

Coincidence: on main diag.

Evals are $\underset{(\text{mult. } 2)}{\lambda_1 = 5}$, $\lambda_2 = 4$

① Find $n=3$ LI evecs of A

$\lambda_1 = 5$

$$[5I - A \mid \vec{0}]$$

$$\vec{x} = t\underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}_{\vec{p}_1} + u\underbrace{\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}}_{\vec{p}_2} \quad \text{Espace is 2-dim.}$$

← LI

$\underline{\lambda_2 = 4}$

$$\left[\,4I - A \,\middle|\, \vec{0}\,\right]$$

$$\vec{x} = t\underbrace{\begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix}}_{\vec{p}_3}$$

② Let $P = [\vec{p}_1, \vec{p}_2 \ \vec{p}_3]$

$$= \begin{bmatrix} 0 & -2 & -\frac{1}{2} \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

③ Let $D = P^{-1}AP$

$$D = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

Cum. Test #12 p. 435

<u>Ex</u> Diagonalize $A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

Only eval is 1.

<u>① Find $n=3$ LI evecs of A</u>

<u>$\lambda_1 = 1$</u>

$$\left[1I - A \,\middle|\, \vec{0} \right]$$

$$\vec{x} = t\underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{\vec{p}_1}$$ Just a 1-dim espace!

We can't get 3 evecs that form a LI set.

We can't get $\vec{p}_2, \vec{p}_3$!!

A is <u>not</u> diag'e.

<u>Ex 8 (p. 400)</u>

$T: \mathbb{R}^3 \to \mathbb{R}^3$
$T(\vec{x}) = A\vec{x}$, where

$$A = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 3 & 1 \\ -3 & 1 & -1 \end{bmatrix}$$

Book: B

Find a basis B' for $\mathbb{R}^3$ such that the matrix for T relative to B' is diagonal.

You diagonalize A.

Think of $\vec{b}_1, \vec{b}_2, \vec{b}_3$ as arrows, as fixed physical entities

$$P = \begin{bmatrix} -1 & 1 & -1 \\ 0 & -1 & 1 \\ 1 & 4 & 1 \end{bmatrix}$$

$$\underbrace{\vec{b}_1 \quad \vec{b}_2 \quad \vec{b}_3}_{B'}$$

Let $D = \underset{B \to B'}{P^{-1}} \underset{T \text{ "in" } B}{A} \underset{B' \to B \text{ (standard)}}{P}$

$$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

is the matrix for T relative to B'.

$\vec{b}_1 \to \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

rel. to new basis / in new coord sys.

what's $D \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$

That reflects the fact that the 1st evec we found is stretched by a fac of 2 under T.

$D \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$

$[\vec{b}_1]_{B'} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

7.3: ORTHOGONALLY DIAGONALIZING SYMMETRIC MATRICES

(A) Symmetric Matrices

A - $n \times n$, real

A is <u>symmetric</u> $\Leftrightarrow A^T = A$
(rows $\leftrightarrow$ cols)

Ex $\begin{bmatrix} 0 & 1 & \boxed{-2} \\ 1 & 3 & 5 \\ \boxed{-2} & 5 & -4 \end{bmatrix}$

Book's term

Real Spectral Theorem

If A is (real) symmetric, then

(1) A is diag'e.
(An eval of (alg.) multiplicity k will have a k-dim espace. You can find n LI evecs.) } All alg. mults. = geom. mults. for all evals.

(2) All evals of A are real.
(The set of evals is called the <u>spectrum</u> of A.)

(B) Orthogonal Matrices

P - $n \times n$

: P is orthogonal (OG)

To find the inverse of an orthogonal matrix, you simply take its ^{T}!

We generally don't want to mess w/ P^{-1}.

$\overset{\text{Definition}}{\Longleftrightarrow}$ P is invertible and $P^{-1} = P^T$ - Nice property if you need to find P^{-1}!

$PP^{-1} = PP^T$

$I = PP^T$ (equivalent)

To show P is OG, Nice!

See Ex 5 (p. 407)

$\overset{\text{Theorem}}{\Longleftrightarrow}$ its column vectors form an orthonormal set, $n \times n$

(pairwise) orthogonal unit vectors

WARNING: An orthogonal matrix is a square matrix with orthonormal cols.

(C) A is orthagonally diagonalizable

$\overset{\text{Def'n}}{\Longleftrightarrow}$ there exists an orthagonal matrix P such that $D = P^{-1}AP$

some diagonal matrix

if P is OG, how can we rewrite $P^{-1}AP$?

Reminder: to invert P, you just take its transpose

$(D = P^{T}AP)$

$\overset{\text{Theorem}}{\Longleftrightarrow}$ A is symmetric.

(D) How do you orthogonally diagonalize a (real) symmetric matrix A?

what else must we do?

Diagonalize A as usual, except you must normalize the $\vec{p}_i$'s.

Read Ex 8 (p. 410)

$$A = \begin{bmatrix} -2 & 2 \\ 2 & 1 \end{bmatrix}$$

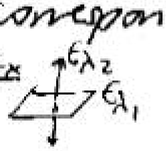
$$\lambda_1 = -3 \longrightarrow \vec{p}_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix} \longrightarrow \vec{u}_1 = \begin{bmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$$

$$\lambda_2 = 2 \longrightarrow \vec{p}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \longrightarrow \vec{u}_2 = \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$$

$P = [\vec{u}_1 \ \vec{u}_2]$

$= \begin{bmatrix} -2/\sqrt{5} & 1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}$

<u>Thm</u>: If A is symmetric, evecs corresponding to different evals are $\perp$. Ex



$\vec{u}_1 \perp \vec{u}_2$

<u>What if an espace has dim ≥ 2?</u>

What can we use to get

Basis $\xrightarrow{\text{Gram-Schmidt}}$ Orthonormal basis

breathe

(Not in this class, at least for a Ch. 7 problem.)

If someone wanted to see e-thing It's easy to invert P – you just take the transpose

Let $D = P^{-1}AP$ or P^TAP } You don't have to work out! Just list the evals in a diagonal matrix like so

Do you have to work out P^TAP?

Then, $D = \begin{bmatrix} \overset{\lambda_1}{-3} & 0 \\ 0 & \underset{\lambda_2}{2} \end{bmatrix}$