CH. 7: EIGENVALUES and EIGENVECTORS

7.1: INTRO

Ⓐ Definitions

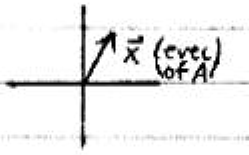
$A - n \times n$

The scalar λ ("lambda") is an <u>eigenvalue</u> (eval) of $A \Leftrightarrow$ there exists $\vec{x} \neq \vec{0}$ such that $A\vec{x} = \lambda\vec{x}$.

$\vec{x}$ is then an <u>eigenvector</u> (evec) of A corresp. to λ.

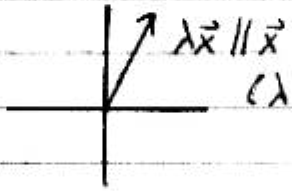
<u>Idea</u>

If we apply



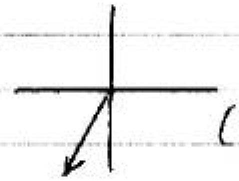
$\xrightarrow{T(\vec{x}) = A\vec{x}}$ scalar multiple of $\vec{x}$

shrinks if $\lambda = \frac{1}{2}$



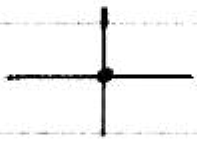
$(\lambda > 0)$

<u>or</u>



$(\lambda < 0)$

<u>or</u>



$(\lambda = 0)$

(B) Verifying Evals, Evecs

6.1.4 · 20f

Ex Verify that $\lambda = 3$ is an eval of $A = \begin{bmatrix} 5 & -3 \\ -4 & 9 \end{bmatrix}$ and that $\vec{x} = (3, 2)$ is a corresp. evec.

Solution

$$A\vec{x} = \begin{bmatrix} 5 & -3 \\ -4 & 9 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 9 \\ 6 \end{bmatrix}$$

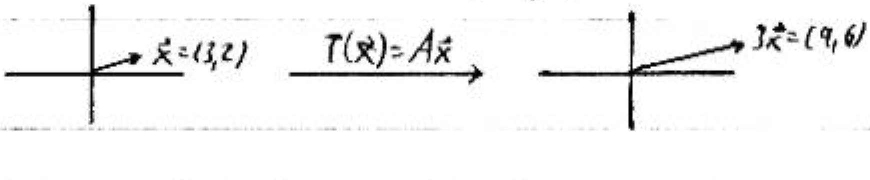
$$\lambda\vec{x} = 3\begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

same

$$= \begin{bmatrix} 9 \\ 6 \end{bmatrix}$$

So, $A\vec{x} = \lambda\vec{x}$ where $\underbrace{\lambda = 3, \vec{x}}_{\text{eval-evec pair}} = (3, 2)$. ✓

3 is a stretching factor.



6.1.6-20F

<u>Ex</u> Is $\vec{x} = (1,-2,1)$ an evec of $A = \begin{bmatrix} 3 & 6 & 7 \\ 3 & 3 & 7 \\ 5 & 6 & 5 \end{bmatrix}$? If so, find its corresp. eval.

$$A\vec{x} = \begin{bmatrix} 3 & 6 & 7 \\ 3 & 3 & 7 \\ 5 & 6 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -2 \\ 4 \\ -2 \end{bmatrix}$$

(Is this a scalar multiple of $\vec{x} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$? YES)

eval-evec pair

$$= \left(-2\vec{x} \quad \text{or}\right) -2 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

$\vec{x}$ <u>is</u> an evec

its corresp. eval is -2

ⓒ Eigenspaces

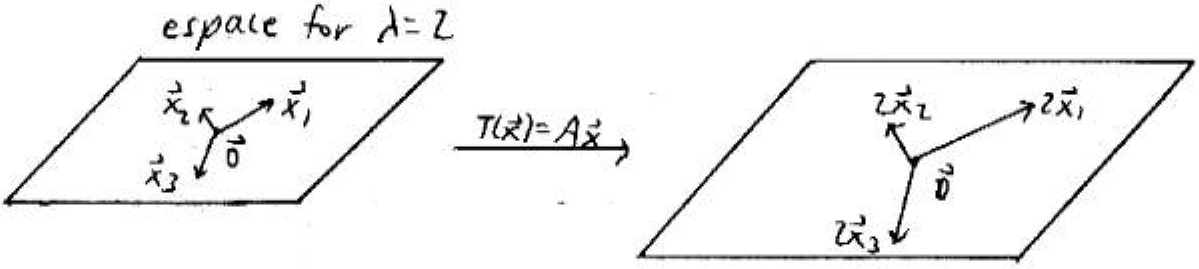
If A $(n \times n)$ has an eval λ, then

$$E_\lambda = \{ \overset{\text{all}}{\text{evecs}} \text{ of } \lambda \} \underbrace{\cup \{\vec{0}\}}_{\text{throw in } \vec{0}}$$

is the eigenspace (espace) of λ.
a subspace of $\mathbb{R}^n$

Ex

What happens to $\vec{x}_1$ under this transf?



Proof: E_λ is a subspace of $\mathbb{R}^n$
$E_\lambda \neq \emptyset$, $E_\lambda \subseteq \mathbb{R}^n$
has $\vec{0}$
Let $\vec{x}_1, \vec{x}_2$ be in E_λ. (They're evecs of λ or $\vec{0}$.) $A\vec{x}_1 = \lambda\vec{x}_1$, $A\vec{x}_2 = \lambda\vec{x}_2$
Show $\vec{x}_1 + \vec{x}_2$ is in E_λ: (i.e., $A(\vec{x}_1 + \vec{x}_2) = \lambda(\vec{x}_1 + \vec{x}_2)$)
$A\vec{x}_1 = \lambda\vec{x}_1$ even if $\vec{x}_1 = \vec{0}$
$$A(\vec{x}_1 + \vec{x}_2) = A\vec{x}_1 + A\vec{x}_2 = \lambda\vec{x}_1 + \lambda\vec{x}_2 = \lambda(\vec{x}_1 + \vec{x}_2)$$
Show $c\vec{x}_1$ is in E_λ: (c is any scalar) (i.e., $A(c\vec{x}_1) = \lambda(c\vec{x}_1)$)
$$A(c\vec{x}_1) = c(A\vec{x}_1) = c(\lambda\vec{x}_1) = \lambda(c\vec{x}_1)$$

① Finding Evals, Evecs

A - $n \times n$

When does $A\vec{x} = \lambda\vec{x}$ have nontrivial sol'ns $\vec{x}$ (evecs)?

$$
\begin{aligned}
& A\vec{x} = \lambda\vec{x} \\
\Leftrightarrow\ & A\vec{x} = \lambda(I\vec{x}) \\
\Leftrightarrow\ & \vec{0} = \lambda I\vec{x} - A\vec{x} \\
\Leftrightarrow\ & \vec{0} = (\lambda I - A)\vec{x}
\end{aligned}
$$

More precisely, $I = I_n$.

or: $A\vec{x} - \lambda I\vec{x} = \vec{0}$

$(A - \lambda I)\vec{x} = \vec{0}$

When does $\underbrace{(\lambda I - A)}_{n \times n}\vec{x} = \vec{0}$ have nontrivial sol'ns $\vec{x}$?

When det = ?

IMT

When $\underbrace{\det(\lambda I - A) = 0}_{\text{i.e., singular/noninvertible}}$ (Some books: $\det(A - \lambda I) = 0$)

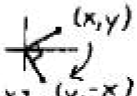
How do we find evals?

Solve $\underbrace{\underbrace{\det(\lambda I - A)}_{\text{characteristic polynomial of } A} = 0}_{\text{characteristic equation of } A}$ for λ.

Meyer 549: If A real skew-sym ($A^T = -A$) $\Rightarrow$ A has pure imag. evals

Maybe no real evals!

Ex $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$



$A\begin{bmatrix} x \\ y \end{bmatrix} = (y, -x)$

No evecs. for real evals.

$\det(\lambda I - A) = |\lambda I - A| = \begin{vmatrix} \lambda & -1 \\ 1 & \lambda \end{vmatrix}$

$= \lambda^2 + 1$ has no real zeros/roots. No real evals!

How do we find the evecs corresp. to λ?

We take the nonzero sol'ns of

$$(\lambda I - A)\vec{x} = \vec{0}$$

Ex Find the evals and corresp. evecs of $A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$.

Do not apply EROs now! EROs do not necessarily preserve evals.

Find evals

Solve $|\lambda I - A| = 0$

$$\left| \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} \right| = 0$$

Here, in principle, you can apply EROs - we're talking about dets.

$$\begin{vmatrix} \lambda-1 & -2 \\ -3 & \lambda-2 \end{vmatrix} = 0$$

Sometimes (n odd?) leading coeff. is +1, not −1 if do $|\lambda I - A|$ vs. $|A - \lambda I|$

$$(\lambda-1)(\lambda-2) - 6 = 0$$

$$\lambda^2 - 3\lambda + 2 - 6 = 0$$

$$\underbrace{\underbrace{\lambda^2 - 3\lambda - 4}_{\text{char. poly. of } A} = 0}_{\text{char. eq. of } A}$$

$$(\lambda-4)(\lambda+1)=0$$ or QF (Quadratic formula)

$$\lambda_1 = 4, \quad \lambda_2 = -1$$

<u>Find the evecs for $\lambda_1 = 4$</u>

Find the nonzero sol'ns of

$$(\lambda I - A)\vec{x} = \vec{0}$$

$$\left[\begin{array}{c|c} 4I - A & \vec{0} \end{array}\right]$$

write 4I 1st

$$\left[\begin{array}{cc|c} 4-1 & 0-2 & 0 \\ 0-3 & 4-2 & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 3 & -2 & 0 \\ -3 & 2 & 0 \end{array}\right]$$

You better have a row of 0s!! [square | $\vec{0}$]

$$\sim \left[\begin{array}{cc|c} 3 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$ ← You better have a row of "0"s!

$$\overset{RRE}{\sim} \left[\begin{array}{cc|c} 1 & -\frac{2}{3} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\begin{cases} x_1 - \frac{2}{3}x_2 = 0 \\ \cancel{0=0} \end{cases}$$

$$\left\{ x_1 = \frac{2}{3}x_2 \right.$$

Let $x_2 = t$

$$\begin{cases} x_1 = \frac{2}{3}t \\ x_2 = t \end{cases}$$

$$\vec{x} = t\begin{bmatrix} 2/3 \\ 1 \end{bmatrix}, \ t \neq 0$$

Shortcut when 1 free var:

$$\frac{2}{3}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 1\begin{bmatrix} -2/3 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

sol'n $\leftrightarrow$ cols from RRE form as weights, we get RHS

($\vec{0}$ isn't an evec)

Find the evecs for $\lambda_2 = -1$

$$\left[(-1)I - A \,\middle|\, \vec{0} \right]$$

$$\left[\begin{array}{cc|c} -1-1 & 0-2 & 0 \\ 0-3 & -1-2 & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} -2 & -2 & 0 \\ -3 & -3 & 0 \end{array}\right]$$

$$\overset{RRE}{\sim} \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$x_1 + x_2 = 0$$

$$\Rightarrow x_1 = -x_2$$

Let $x_2 = t$

$$\begin{cases} x_1 = -t \\ x_2 = t \end{cases}$$

$$\vec{x} = t\begin{bmatrix} -1 \\ 1 \end{bmatrix}, \ t \neq 0$$

Shortcut:
$(-1)\begin{bmatrix} 0 \\ 0 \end{bmatrix} + (1)\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

In general, the char. poly. is n^{th}-degree in λ.
A can have at most n real distinct evals.

If $n \geq 3$, I will give you the evals, unless...
see my Handout

Ⓔ Triangular Matrices

Their evals are the
main diagonal entries.

Ex
$$A = \begin{bmatrix} 1 & 0 & 0 \\ 4 & -2 & 0 \\ 5 & 9 & 0 \end{bmatrix}$$

is lower triangular.

Evals: $1, -2, 0$

Why?
$$A = \begin{bmatrix} d_1 & & O \\ & d_2 & \\ & & \ddots \\ & & & d_n \end{bmatrix}$$

$$|\lambda I - A| = \begin{vmatrix} \lambda - d_1 & & O \\ & \lambda - d_2 & \\ & & \ddots \\ & & & \lambda - d_n \end{vmatrix}$$

7.1.10

$$= (\lambda - d_1)(\lambda - d_2)\cdots(\lambda - d_n) \stackrel{\text{set}}{=} 0$$

$$\lambda = d_1,\ \lambda = d_2, \ldots, \lambda = d_n$$

Similarly for

Ⓔ More Exs.

8.1.20 **Ex** $\lambda = 7$ is one of the evals of

$$A = \begin{bmatrix} 7 & -4 & 4 \\ -4 & 5 & 0 \\ 4 & 0 & 9 \end{bmatrix}$$

Find the evecs corresp. to $\lambda = 7$.

$$\left[\underset{7}{\lambda} I - A \;\middle|\; \begin{matrix} 0 \\ 0 \\ 0 \end{matrix}\right]$$

$$\left[\begin{matrix} 7-7 & 0-(-4) & 0-4 \\ 0-(-4) & 7-5 & 0-0 \\ 0-4 & 0-0 & 7-9 \end{matrix} \;\middle|\; \begin{matrix} 0 \\ 0 \\ 0 \end{matrix}\right]$$

$$\left[\begin{matrix} 0 & 4 & -4 \\ 4 & 2 & 0 \\ -4 & 0 & -2 \end{matrix} \;\middle|\; \begin{matrix} 0 \\ 0 \\ 0 \end{matrix}\right]$$

$$\overset{RRE}{\sim} \left[\begin{matrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{matrix} \;\middle|\; \begin{matrix} 0 \\ 0 \\ 0 \end{matrix}\right]$$

x_3 free

$$\begin{cases} x_1 + \frac{1}{2}x_3 = 0 \\ x_2 - x_3 = 0 \end{cases}$$

$$\begin{cases} x_1 = -\frac{1}{2}x_3 \\ x_2 = x_3 \end{cases}$$

Let $x_3 = t$

$$\begin{cases} x_1 = -\frac{1}{2}t \\ x_2 = t \\ x_3 = t \end{cases}$$

$-\frac{1}{2}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \ldots$

$$\vec{x} = t\begin{bmatrix} -1/2 \\ 1 \\ 1 \end{bmatrix}, t \neq 0$$ Shortcut

(Espaces don't have to be 1-dim!!)

Ex 6 (p.386)

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 5 & -10 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 3 \end{bmatrix}$$

Find evals (I'd give them)

Solve $|\lambda I - A| = 0$

$$(\lambda-1)^{2}(\lambda-2)(\lambda-3)=0$$

$\lambda_1=1$ has (algebraic) multiplicity 2
$\lambda_2=2$ " " 1
$\lambda_3=3$ " " 1

The espace for $\lambda_1=1$ could be 1-dim or 2-dim

When you solve

$$\left[(1)I-A \;\middle|\; \vec{0}\right]$$

$$\vec{x}=s\begin{bmatrix}0\\1\\0\\0\end{bmatrix}+t\begin{bmatrix}-2\\0\\2\\1\end{bmatrix}$$

The espace for $\lambda_1=1$ is 2-dim.

form a basis (for the espace for $\lambda_1=1$)

<u>Ex</u>

$$\begin{bmatrix}1 & & & \\ & 4 & & \\ & & 4 & \\ 0 & & & 4\end{bmatrix}$$

Anton 353
3 = algebraic mult.
dim(espace) = geometric mult.
Don't write-confusing?
What could dim of espace be?

$\lambda=4$ has (alg) multiplicity 3
Its espace can have dim. 1, 2, or 3.
geometric mult'y.