7.2: DIAGONALIZATION

A - $n \times n$

(A) Definition

A is diagonalizable (diag'e) $\leftrightarrow$
A is similar to a diagonal matrix

i.e., there exists an invertible $n \times n$ matrix P such that $P^{-1}AP$ is diagonal.

a matrix is similar to itself

Ex If A is diagonal, $I^{-1}AI = A$ is diagonal, so A is diag'e.

Ex Verify that A is diag'e by showing that $P^{-1}AP$ is diagonal.

p. 450 #1

$$A = \begin{bmatrix} 2 & 1 \\ 5 & -2 \end{bmatrix} \quad P = \begin{bmatrix} 1 & 1 \\ -5 & 1 \end{bmatrix}$$

matrix mult. is assoc.

$$P^{-1}AP = \underbrace{\begin{bmatrix} 1 & 1 \\ -5 & 1 \end{bmatrix}^{-1}}_{\frac{1}{\det(P)}\begin{bmatrix} 1 & -1 \\ 5 & 1 \end{bmatrix}} \underbrace{\begin{bmatrix} 2 & 1 \\ 5 & -2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -5 & 1 \end{bmatrix}}_{\begin{bmatrix} -3 & 3 \\ 15 & 3 \end{bmatrix}}$$

flip signs

switch

$$= \frac{1}{6}\begin{bmatrix} 1 & -1 \\ 5 & 1 \end{bmatrix}\begin{bmatrix} -3 & 3 \\ 15 & 3 \end{bmatrix}$$

$$= \frac{1}{6}\begin{bmatrix} -18 & 0 \\ 0 & 18 \end{bmatrix}$$

$$= \begin{bmatrix} -3 & 0 \\ 0 & 3 \end{bmatrix}$$ diagonal ✓
So, A is diag'e.

evals are invariant under sim.

pf p.393
$|\lambda I - P^{-1}AP|$
$= |P^{-1}\lambda I P - P^{-1}AP|$
$= |P^{-1}(\lambda I - A)P|$
$= |P^{-1}||\lambda I - A||P|$

evals are what?

Ⓑ <u>Similar matrices have the same evals.</u>

If $D = P^{-1}AP$ ← A and D are similar

D: diagonal: its evals are on its main diagonal

A: has the same evals

<u>Ex</u> In Ⓐ

$$\left(A = \begin{bmatrix} 2 & 1 \\ 5 & -2 \end{bmatrix} \text{ is similar to} \right)$$

$$A \sim D = \begin{bmatrix} -3 & 0 \\ 0 & 3 \end{bmatrix}$$

-3 and 3 are the evals of D <u>and</u> of A

Ⓒ When is A diag'e?

if you have n LI vecs in an n-dim space, they also span

$\Leftrightarrow$ there exist n LI evecs of A
(i.e., an evec-basis for $\mathbb{R}^n$)

Ⓓ How do you diagonalize A?

We need P, diagonal D such that

$$D = P^{-1}AP$$

① Find n LI evecs of A:

$\vec{p}_1$ (w/ corresp. eval λ_1)
$\vec{p}_2$ λ_2
$\vdots$
$\vec{p}_n$ λ_n ← maybe duplicates

If you can't, then A is not diag'e.
If you can ...

② Let $P = [\vec{p}_1 \; \vec{p}_2 \; \cdots \; \vec{p}_n]$

cols. are the n LI evecs

③ Let $D = P^{-1}AP$ (D, A similar $\rightarrow$ same evals)

Then, $D = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots & \\ & & & \lambda_n \end{bmatrix}$

order corresponds to order of evecs in P

Then, we'll do some examples.

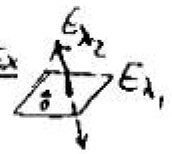
Ⓔ LI Theorems

"Different espaces are LI"

Evecs corresponding to different evals (espaces) form a LI set.

If different $\begin{matrix} \lambda_1 \to \vec{p}_1 \\ \lambda_k \to \vec{p}_k \end{matrix}\Bigg\}$ then, LI

Each eval has its own espace, with only $\vec{0}$ in common.



Each eval has a 1-dim. espace.

If A has n different real evals, then A is <u>guaranteed</u> to be diag'e, since you can find n LI evecs.

(F) Examples

Cum. Test p. 435, #11

Ex Diagonalize $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 3 & 1 \\ 0 & -3 & -1 \end{bmatrix}$

Evals are: 0, 1, 2 (given).

A is 3×3. A has $n=3$ different evals, so we know A is diag'e.

① Find $n=3$ LI evecs of A

$\lambda_1 = 0$

What are its evecs?

$$[\cancel{0I} - A \,|\, \vec{0}\,]$$

$$\vdots$$

$$\vec{x} = t\begin{bmatrix} -1 \\ -1 \\ 3 \end{bmatrix},\ t \neq 0$$

Espace for 0 is 1-dim.

Can grab any vector from this 1-D espace

Let $\vec{p}_1 = \begin{bmatrix} -1 \\ -1 \\ 3 \end{bmatrix}$ $\left(\text{or } \begin{bmatrix} -2 \\ -2 \\ 6 \end{bmatrix}, \text{etc.}\right)$

$\underline{\lambda_2 = 1}$

$$\left[1I - A \,\middle|\, \vec{0} \right]$$

$$\vdots$$

$$\vec{x} = t \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{\vec{p}_2}, \quad t \neq 0$$

$\underline{\lambda_3 = 2}$

$$\left[2I - A \,\middle|\, \vec{0} \right]$$

$$\vdots$$

$$\vec{x} = t \underbrace{\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}}_{\vec{p}_3}, \quad t \neq 0$$

② Let $P = [\vec{p}_1 \ \ \vec{p}_2 \ \ \vec{p}_3]$

$$= \begin{bmatrix} -1 & 1 & 1 \\ -1 & 0 & 1 \\ 3 & 0 & -1 \end{bmatrix}$$

③ Let $D = P^{-1}AP$

Don't even have to find P^{-1} in HW, you're asked to verify

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

If you reorder $\vec{p}_i$'s you must reorder λ_i's

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Bottom line: Give P, D

Cn 6.3.6

__Ex__ Diagonalize $A = \begin{bmatrix} 4 & 0 & -2 \\ 2 & 5 & 4 \\ 0 & 0 & 5 \end{bmatrix}$

Coincidence: on main diag.

Evals are $\underset{(\text{mult. } 2)}{\lambda_1 = 5}$, $\lambda_2 = 4$

__① Find $n = 3$ LI evecs of A__

__$\lambda_1 = 5$__

$$[5I - A \mid \vec{0}]$$

$$\vec{x} = t\underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}_{\vec{p}_1} + u\underbrace{\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}}_{\vec{p}_2}$$ Espace is 2-dim.

← LI

$\underline{\lambda_2 = 4}$

$$\left[4I - A \mid \vec{0} \right]$$

$$\vec{x} = t \underbrace{\begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix}}_{\vec{p}_3}$$

② Let $P = [\vec{p}_1 \ \vec{p}_2 \ \vec{p}_3]$

$$= \begin{bmatrix} 0 & -2 & -\frac{1}{2} \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

③ Let $D = P^{-1}AP$

$$D = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

Cum. Test #12 p. 435

<u>Ex</u> Diagonalize $A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

Only eval is 1.

<u>① Find $n=3$ LI evecs of A</u>

<u>$\lambda_1 = 1$</u>

$$\left[1I - A \mid \vec{0} \right]$$

$$\vec{x} = t \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{\vec{p}_1}$$ Just a 1-dim espace!

We can't get 3 evecs that form a LI set.

We can't get $\vec{p}_2, \vec{p}_3$!!

A is <u>not</u> diag'e.

<u>Ex 8 (p. 400)</u>

$T: \mathbb{R}^3 \to \mathbb{R}^3$
$T(\vec{x}) = A\vec{x}$, where

$$A = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 3 & 1 \\ -3 & 1 & -1 \end{bmatrix}$$

7.2.10

Book: B

Find a basis B' for $\mathbb{R}^3$ such that the matrix for T relative to B' is diagonal.

You diagonalize A.

Think of $\vec{b}_1, \vec{b}_2, \vec{b}_3$ as arrows, as fixed physical entities

$$P = \begin{bmatrix} -1 & 1 & -1 \\ 0 & -1 & 1 \\ 1 & 4 & 1 \end{bmatrix}$$

$$\underbrace{\vec{b}_1 \quad \vec{b}_2 \quad \vec{b}_3}_{B'}$$

Let $D = \underset{B \to B'}{P^{-1}} \; \underset{T \text{ "in" } B}{A} \; \underset{B' \to B \text{ (standard)}}{P}$

$\vec{b}_1 \to \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

rel. to new basis / in new coord sys.

what's $D\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$

That reflects the fact that the 1st evec we found is stretched by a fac of 2 under T.

$$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

is the matrix for T relative to B'.

$D\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$

$[\vec{b}_1]_{B'} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$