

CH. 8: COMPLEX VECTOR SPACESOnly a few  
we'll do!!!8.1: COMPLEX #s

(A) i

$$\begin{aligned} i &= \sqrt{-1} && (\text{imaginary unit}) \\ i^2 &= -1 && (i \text{ solves } x^2 = -1) \\ && x^2 + 1 = 0 \\ && (-i, \text{ also}) \end{aligned}$$

(B) Simplifying Radicals

Ex Simplify  $\sqrt{-16}$ 

$\sqrt{-16}$  - that counts  
 $\{1\}(-)$

$$\sqrt{-16} = i\sqrt{16} = 4i$$

-1 factor  
 comes out as i if this  
 is +

$$\text{Ex } \sqrt{-17} = i\sqrt{17} \text{ or } \sqrt{17}i^{1 \text{ step!}}$$

$$\text{Ex } \sqrt{-18} = i\sqrt{18} \quad (\text{largest perfect square that divides 18 is 9})$$

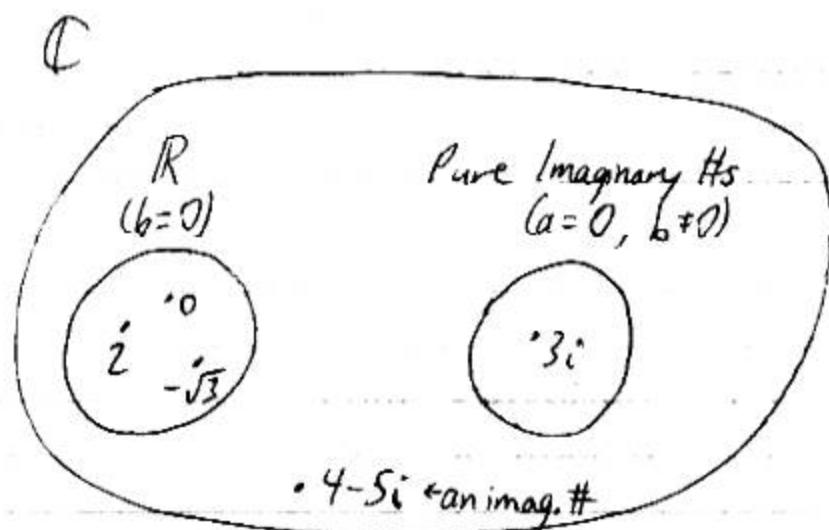
$$= 3i\sqrt{2} \quad (9 \text{ comes out as 3.})$$

(C) Complex #s

Standard form:  $a + bi$  ( $a, b$  real)

$\begin{matrix} \uparrow & \downarrow \\ \text{real part} & \text{imaginary part} \end{matrix}$

$\mathbb{C} = \text{set of all complex } \#s$   
 includes  $\mathbb{R}$  ( $\mathbb{R} \subseteq \mathbb{C}$ )



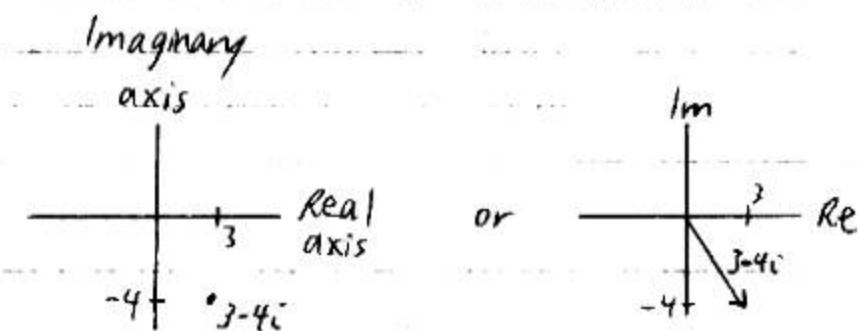
Treating  $\mathbb{C}$  as a real vector space,  $\dim(\mathbb{C}) = 2$ .  
 If you take all poss LCS of  $1, i$ , you get  $\mathbb{C}$ .  
 Usual basis:  $\{1, i\}$   
 $\mathbb{R}$  is a 1-dim subspace of  $\mathbb{C}$ .  
 $\text{span}\{1\}$

## ① The Complex Plane

Real Line



Now,



We plot  $a+bi$  as the point  $(a, b)$   
 or the vector  $(a, b)$



E) +, -, ×

① Simplify all radicals.

② Think of  $i$  as "x".

$+, -, \times$  as usual.

Use  $i^2 = -1$ . (No  $i^2, i^3, \dots$  in answer.)

③ Simplify  $\rightarrow a+bi$

$$\text{Ex } \sqrt{-9}\sqrt{-4}$$

Must do ① 1st!

$\sqrt{a}\sqrt{b} \neq \sqrt{ab}$   
if  $a, b < 0$

$$= (3i)(2i)$$

$$= 6i^2$$

$$= 6(-1)$$

$$= -6 \quad \text{Note: } \sqrt{-9}\sqrt{-4} \neq \sqrt{36}$$

Ex Find the difference of  $4+4i, 6+5i$

$$\begin{aligned} & (4+4i) - (6+5i) \\ &= 4+4i - 6-5i \\ &= \cancel{4} \cancel{-6} + \cancel{4i} \cancel{-5i} \\ &= -2-i \end{aligned}$$

$$\text{Ex } (2+3i)(4+i) \stackrel{\text{Exp}}{=} 8+2i+12i+3i^2$$

$$= 8+14i+3(-1)$$

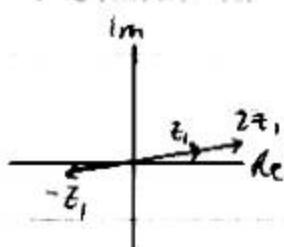
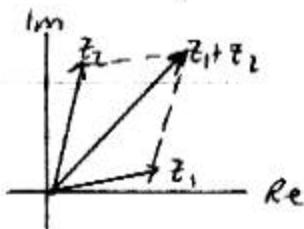
$$= 8+14i-3$$

$$= \boxed{5+14i}$$

Up to 11

$z_1, z_2$  are complex #s.

usual  
graphical  
depiction  
of vector +  
in  $\mathbb{R}^n$



Vector + and scalar mult. of  
these vectors work as in  $\mathbb{R}^2$ .  
 $\mathbb{C} \cong \mathbb{R}^2$

Up to 31

## F Zeros of a Polynomial

Find the zeros of  $p(x) = x^2 + x + 1$ .  
 i.e., Solve  $x^2 + x + 1 = 0$ .

$$\begin{aligned}
 x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} && \leftarrow \text{sols of } ax^2 + bx + c = 0 \\
 &= \frac{-1 \pm \sqrt{(1)^2 - 4(1)(1)}}{2(1)} \\
 &= \frac{-1 \pm \sqrt{-3}}{2} \\
 &= \frac{-1 \pm \sqrt{3}i}{2} && \text{complex conjugate pair}
 \end{aligned}$$

## ⑥ Complex Matrices

Ex  $A = \begin{bmatrix} 3-i & i \\ 1+2i & 2 \end{bmatrix}$ ,  $B = \begin{bmatrix} 2i & 4 \\ 1-i & 3i \end{bmatrix}$

$$3(A+B) = 3 \begin{bmatrix} (3-i)+(2i) & i+4 \\ (1+2i)+(1-i) & 2+3i \end{bmatrix}$$

$$= 3 \begin{bmatrix} 3+i & 4+i \\ 2+i & 2+3i \end{bmatrix}$$

$$= \begin{bmatrix} 9+3i & 12+3i \\ 6+3i & 6+9i \end{bmatrix}$$

$$AB = \begin{bmatrix} 3-i & i \\ 1+2i & 2 \end{bmatrix} \begin{bmatrix} 2i & 4 \\ 1-i & 3i \end{bmatrix}$$

$$= \begin{bmatrix} (3-i)(2i) + (i)(1-i) & (3-i)(4) + (i)(3i) \\ (1+2i)(2i) + (2)(1-i) & (1+2i)(4) + (2)(3i) \end{bmatrix}$$

Don't think "·"! See 8.4.9.

$$\begin{aligned} &= \begin{pmatrix} 6i - 2i^2 + i - i^2 & 12 - 4i + 3i^2 \\ 2i + 4i^2 + 2 - 2i & 4 + 8i + 6i \end{pmatrix} \\ &= \begin{pmatrix} 6i + 2 + i - (-1) & 12 - 4i - 3 \\ 2i - 4 + 2 - 2i & 4 + 14i \end{pmatrix} \end{aligned}$$

$$= \begin{bmatrix} 3+7i & 9-4i \\ -2 & 4+14i \end{bmatrix}$$

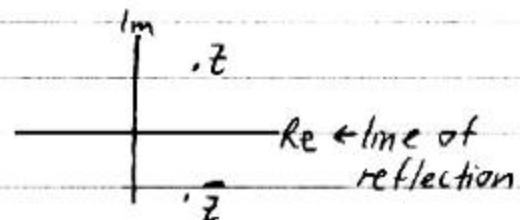
$$\begin{aligned}|A| &= \begin{vmatrix} 3-i & i \\ 1+2i & 2 \end{vmatrix} \\&= (3-i)(2) - (i)(1+2i) \\&= 6-2i - i - 2i^2 \\&= 6-3i + 2 \\&= \underline{\underline{(8-3i)}} \neq 0 \Rightarrow \text{A is inv.}\end{aligned}$$

8.2: CONJUGATES,  $\div$ (A) Complex Conjugates (CCs)

A pair of CCs:

$$z = a + bi \quad \text{change the sign}$$

$$\bar{z} = a - bi \quad \text{of the imag. part}$$



Ex  $z = 2 + 5i \Rightarrow \bar{z} = 2 - 5i$

Ex  $z = 5 \Rightarrow \bar{z} = 5$

Ex  $z = -4i \Rightarrow \bar{z} = 4i$

If  $z = a + bi$ , then  $z\bar{z} = (a + bi)(a - bi) = a^2 - b^2i^2 = a^2 + b^2$

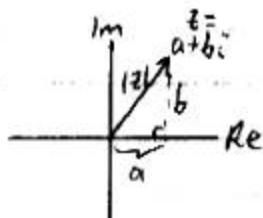
Ex  $z = -1 + 2i \quad \leftarrow a = -1, b = 2$   
 $\bar{z} = -1 - 2i$

$$\begin{aligned} z\bar{z} &= (-1 + 2i)(-1 - 2i) \\ &= (-1)^2 - (2i)^2 \\ &= 1 - 4i^2 \\ &= 1 + 4 \quad \leftarrow a^2 + b^2 \\ &= 5 \end{aligned}$$

$\begin{aligned} z\bar{z} &= (a + bi)(a - bi) \\ &= a^2 - b^2i^2 \\ &= a^2 + b^2 \end{aligned}$	$\begin{aligned} z\bar{z} &= (-1)^2 + (2)^2 \\ &= 1 + 4 \\ &= 5 \end{aligned}$
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### B) Modulus

$|z| = \text{the modulus (or absolute value) of } z$   
 $= \text{length of vector (or distance from origin)}$   
 $= \sqrt{a^2 + b^2}$ , where  $z = a + bi$



we've seen  
this # before

Ex If  $z = -1 + 2i$ ,  
then  $|z| = \sqrt{(-1)^2 + (2)^2} = \sqrt{5}$



$$|z|^2 = z\bar{z}$$

we've seen  
this # before

Ex If  $z = -1 + 2i$ , then  $z\bar{z} = 5$   
 $|z|^2 = (\sqrt{5})^2 = 5$

$$|z_1 z_2| = |z_1| |z_2|$$

$$\neq 0 \rightarrow \left| \frac{z_1}{z_2} \right| = \frac{|z_1|}{|z_2|} \quad (, z_2 \neq 0)$$

C:

Goal: Express  $\frac{z_1}{z_2}$  as  $x+yi$ .

in principle,  
the denominator  
can't be real  
at end

We rationalize ("real"-ize) the denominator.

① Simplify radicals

② If the  $D$  has form  $bi$ ,  
multiply  $N$  and  $D$   
by  $i$ . (or  $-i$ )

If the  $D$  has form  $a+bi$ ,  
multiply  $N$  and  $D$   
by the (complex conjugate) CC  
of the  $D$ .

③ Simplify  $\rightarrow x+yi$  form

$$\left. \begin{aligned} \text{Ex } \frac{4-i}{3i} \cdot \frac{i}{i} &= \frac{4i - i^2}{3i^2} \\ &= \frac{4i - (-1)}{3(-1)} \\ &= \frac{4i + 1}{-3} \\ &= -\frac{1}{3} - \frac{4}{3}i \\ &= \boxed{-\frac{1}{3} - \frac{4}{3}i} \end{aligned} \right\} \quad \left. \begin{aligned} \text{Ex } \frac{(4-i)(-i)}{(3i)(-i)} &= \frac{-4i + i^2}{-3i^2} \\ &= \frac{-4i - 1}{3} \\ &= -\frac{1}{3} - \frac{4}{3}i \end{aligned} \right\}^{(1)}$$

8.2.4

$$\begin{aligned}
 \text{Ex } \frac{3-\sqrt{-16}}{4+\sqrt{-9}} &= \frac{3-4i}{4+3i} \cdot \frac{4-3i}{4-3i} \quad \left[ \begin{array}{l} \text{CC of} \\ D \end{array} \right] \\
 &= \frac{(3-4i)(4-3i)}{(4)^2 + (3)^2} \quad \left[ \begin{array}{l} \text{FOIL} \\ z\bar{z}=a^2+b^2 \end{array} \right] \\
 &= \frac{12-9i-16i+12i^2}{25} \\
 &= \frac{12-25i-12}{25} \\
 &= \boxed{-i \text{ or } 0-i}
 \end{aligned}$$

④  $A^{-1}$   
complex

$$\text{Ex } A = \begin{bmatrix} 3-i & i \\ 1+2i & 2 \end{bmatrix}$$

Find  $A^{-1}$ .

Solution

We've found  
the det  
already

$$\begin{aligned}
 |A| &= (3-i)(2) - (i)(1+2i) \\
 &= 8-3i
 \end{aligned}$$

$|A| \neq 0$ , so  $A$  is invertible

$A \in 2 \times 2$ , so

$$A^{-1} = \frac{1}{|A|} \begin{bmatrix} 2 & -i \\ -1-2i & 3-i \end{bmatrix} \quad \left[ \begin{array}{l} \text{switch} \\ \text{rows} \end{array} \right]$$

$$= \frac{1}{8-3i} \begin{bmatrix} 2 & -i \\ -1-2i & 3-i \end{bmatrix} \quad (\text{not final!})$$

$\underbrace{\phantom{0}}$   
rationalize  
the D

$$= \frac{1}{8-3i} \cdot \frac{8+3i}{8+3i} \begin{bmatrix} 2 & -i \\ -1-2i & 3-i \end{bmatrix}$$

$\underbrace{8+3i}_{(8)^2+(3)^2}$

$$= \frac{8+3i}{73}$$

$$= \frac{8+3i}{73} \begin{bmatrix} 2 & -i \\ -1-2i & 3-i \end{bmatrix}$$

Multiply through  $(8+3i)$ .

$$= \frac{1}{73} \begin{bmatrix} 2(8+3i) & -i(8+3i) \\ (-1-2i)(8+3i) & (3-i)(8+3i) \end{bmatrix}$$

$$\left( = \frac{1}{73} \begin{bmatrix} 16+6i & -8i-3i^2 \\ -8-3i-16i-6i^2 & 24+9i-8i-3i^2 \end{bmatrix} \right)$$

$$= \frac{1}{73} \begin{bmatrix} 16+6i & 3-8i \\ -2-19i & 27+i \end{bmatrix} \quad \text{OK}$$

$$\left( = \begin{bmatrix} \frac{16}{73} + \frac{6}{73}i & \frac{3}{73} - \frac{8}{73}i \\ -\frac{2}{73} - \frac{19}{73}i & \frac{27}{73} + \frac{1}{73}i \end{bmatrix} \quad \begin{array}{l} \text{each entry in standard form} \\ \text{Final} \end{array} \right)$$

$$AA^{-1} = I_2 \checkmark$$

Book 1)  
continues  
on this

8.3.1

8.3

The 4 4<sup>th</sup> roots of 1



Ch. 7 HW  
Weierstrass  
w/complex  
Hg

$$\begin{aligned} D &= P^{-1} A^{\text{regular } n\text{-gon}} P \\ \Rightarrow A^{100} &= P D^{100} P^{-1} \end{aligned}$$

## 8.4: COMPLEX VECTOR SPACES

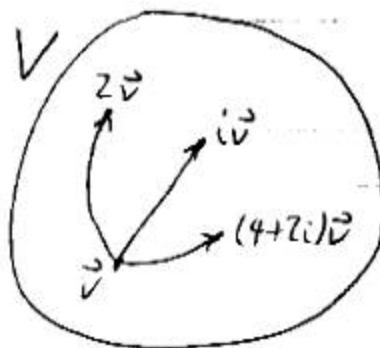
### (A) Complex VSs

The set of scalars is  $\mathbb{C}$ , not just  $\mathbb{R}$ .

If  $\{\vec{b}_1, \vec{b}_2, \dots, \vec{b}_n\}$  is a basis for a complex VS,  $V$ , then

$$V = \{c_1 \vec{b}_1 + c_2 \vec{b}_2 + \dots + c_n \vec{b}_n\}$$

" $c_i$ 's are complex #'s}



### Closure of $V$ under scalar mult.

If  $\vec{v}$  is in  $V$  and if  $c$  is any complex # (maybe real), then  $c\vec{v}$  is in  $V$ .

B)  $\mathbb{C}^n$ 

$$\mathbb{C}^n = \{(z_1, z_2, \dots, z_n) \mid "z_i's \text{ are complex } \#s\}$$

$$= \left\{ \underbrace{(a_1 + b_1 i, \dots, a_n + b_n i)}_{z_1, \dots, z_n} \mid "a_k's, "b_k's \text{ are real } \#s \right\}$$

Ex Let  $\vec{v} = \begin{bmatrix} z_1 \\ 4+i \end{bmatrix}$ ,  $\vec{w} = \begin{bmatrix} -3 \\ 2-z_1 \end{bmatrix}$

$\vec{v}, \vec{w}$  are in  $\mathbb{C}^2$ .  
Find  $\underbrace{2\vec{v} + (1+3i)\vec{w}}_{\text{a linear combo}}$ .

of  $\vec{v}, \vec{w}$

Solution

$$2 \begin{bmatrix} z_1 \\ 4+i \end{bmatrix} + (1+3i) \begin{bmatrix} -3 \\ 2-z_1 \end{bmatrix}$$

$$= \begin{bmatrix} 2(z_1) \\ 2(4+i) \end{bmatrix} + \begin{bmatrix} (1+3i)(-3) \\ (1+3i)(2-z_1) \end{bmatrix}$$

$$\begin{aligned}
 &= \begin{bmatrix} 4i \\ 8+2i \end{bmatrix} + \underbrace{\begin{bmatrix} -3-9i \\ 2-2z_1+6i-6i^2 \end{bmatrix}}_{\begin{bmatrix} -3-9i \\ 2+4i+6 \end{bmatrix}} \\
 &= \begin{bmatrix} -3-9i \\ 8+4i \end{bmatrix}
 \end{aligned}$$

$$= \begin{bmatrix} 4i \\ 8+2i \end{bmatrix} + \begin{bmatrix} -3-9i \\ 8+4i \end{bmatrix}$$

$$= \begin{bmatrix} 4i - 3 - 9i \\ 8+2i + 8+4i \end{bmatrix}$$

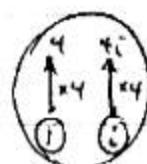
$$= \begin{bmatrix} -3-5i \\ 16+6i \end{bmatrix}$$

### ③ $\mathbb{C}$ : Two Perspectives

As a real VS,  $\mathbb{C}$  has  $\dim = 2$ .

Usual basis =  $\{1, i\}$

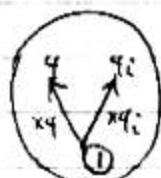
$$\mathbb{C} = \{c_1(1) + c_2(i) \mid c_1, c_2 \text{ are real } \#s\}$$



As a complex VS,  $\mathbb{C}$  has  $\dim = 1$ .

Standard basis =  $\{1\}$

$$\mathbb{C} = \{c(1) \mid c \text{ is a complex } \#\}$$



## ① Bases for $\mathbb{C}^n$

As a real  
vec space,  
 $\dim \mathbb{C}^n$

As a complex VS,  
 $\mathbb{C}^n$  has  $\dim = n$ .  
Standard basis:

same as for  
 $\mathbb{R}^n$ , as real vs

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \right\}$$

$$\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n$$

$$\mathbb{C}^n = \text{Span}(\{\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n\})$$

$$= \{c_1 \vec{e}_1 + c_2 \vec{e}_2 + \dots + c_n \vec{e}_n \mid "c_i's are complex #'s\}$$

Ex Basis for  $\mathbb{C}^2$ : As a complex VS:  $\{[1], [i]\}$ , As a real VS:  $\{[1], [i], [0], [0]\}$   
As a complex VS, Any set of  $n$  LI vectors in  $\mathbb{C}^n$   
is a basis for  $\mathbb{C}^n$ .

$n$  vectors

LI  
span  $\mathbb{C}^n$  ↗ have both  
or neither

Ex Is  $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ i \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1+i \end{bmatrix} \right\}$

a basis for  $\mathbb{C}^3$ ?

6.6 weeks: Anton Ped  
p. 499  
Why not a general  
matrix method?  
Complex #'s  
and Gaussian  
elimination don't  
mix well -  
by complex #'s  
is nontrivial

(Method 1 (lucky!))

$$[\vec{v}_1 \quad \vec{v}_2 \quad \vec{v}_3]$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & i & 2i \\ 0 & 0 & 1+i \end{bmatrix}$$

row-echelon shape!

Full "set" of 3  $\Rightarrow \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$  is a basis ✓

Also,  $\det(A) \neq 0$ .

Gaussian elimination (is often ugly!)

- can use complex #s to rescale, replace.  
 $(\neq 0)$

Method 2 (Book's approach)

$\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$  is a basis for  $\mathbb{C}^3$

$\Leftrightarrow c_1 \vec{v}_1 + c_2 \vec{v}_2 + c_3 \vec{v}_3 = \vec{0}$  has only  
 $c_1 = c_2 = c_3 = 0$  as a sol'n.

$$c_1 \vec{v}_1 + c_2 \vec{v}_2 + c_3 \vec{v}_3$$

$$= c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ i \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 2i \\ 1+i \end{bmatrix}$$

$$= \begin{bmatrix} c_1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ c_2 i \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 2c_3 i \\ c_3(1+i) \end{bmatrix}$$

$$= \begin{bmatrix} (c_1 + 2c_3)i \\ c_3(1+i) \\ 0 \end{bmatrix} \stackrel{\text{set}}{=} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left\{ \begin{array}{l} c_1 = 0 \\ (c_2 + 2c_3)_i = 0 \\ c_3(1+i) = 0 \end{array} \right. \xrightarrow{\text{#}} c_3 = 0$$

$\star: z_1, z_2 = 0 \leftrightarrow z_1 = 0 \text{ or } z_2 = 0$

$$c_3 = 0, \text{ so } (c_2 + 2c_3)_i = 0$$

$$\rightarrow c_2 i = 0 \rightarrow c_2 = 0$$

$$c_1 = c_2 = c_3 = 0 \quad \checkmark$$

$\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$  is a basis for  $\mathbb{C}^3$ .

Ex Is  $\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} i \\ i \\ 0 \end{bmatrix}, \begin{bmatrix} i \\ i \\ i \end{bmatrix} \right\}$

a basis for  $\mathbb{C}^3$ ?

(Method 1 (lucky!)) (NO)  $i\vec{v}_1 + \vec{v}_2 = \vec{v}_3$

$$[\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3]$$

$$\begin{bmatrix} 0 & i & i \\ 0 & i & i \\ 1 & 0 & i \end{bmatrix} \xleftarrow{\text{same, so det=0}} \text{NO}$$

$$R_1 \leftrightarrow R_3$$

6 elem.  
works ok  
here.

There's a  
dependency  
relation

$$\begin{bmatrix} 1 & 0 & i \\ 0 & 1 & i \\ 0 & i & 1 \end{bmatrix}$$

$$R_3 - R_2 \rightarrow R_3$$

$$\begin{bmatrix} 0 & 0 & i \\ 0 & 0 & i \\ 0 & 0 & 0 \end{bmatrix}$$

Only 2 ~~PP~~, so ~~NO~~  
 $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$  is not a basis for  $\mathbb{C}^3$ .

$$\det(A) = 0$$

$$(1) \left| \begin{array}{ccc} 1 & 0 & i \\ 0 & 1 & i \\ 0 & i & 1 \end{array} \right|$$

### Method 2

$$c_1 \vec{v}_1 + c_2 \vec{v}_2 + c_3 \vec{v}_3$$

$$= c_1 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} i \\ 0 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} i \\ 1 \\ i \end{bmatrix}$$

$$= \begin{bmatrix} c_2 i + c_3 i \\ c_2 i + c_3 i \\ c_1 + c_3 i \end{bmatrix}$$

$$= \begin{bmatrix} (c_2 + c_3)i \\ (c_2 + c_3)i \\ c_1 + c_3 i \end{bmatrix} \stackrel{\text{set}}{=} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{cases} (c_2 + c_3)i = 0 \\ (c_2 + c_3)i = 0 \\ c_1 + c_3 i = 0 \end{cases} \rightarrow \begin{cases} c_2 + c_3 = 0 \\ c_2 + c_3 = 0 \\ c_1 + c_3 = 0 \end{cases} \quad \begin{matrix} \text{same} \\ \text{eq!} \end{matrix}$$

2 eqs. in  
3 vars.

$$\begin{cases} c_2 + c_3 = 0 \\ c_1 + c_3 i = 0 \end{cases}$$

$c_3$  is free (any complex #)

$$\begin{cases} c_2 = -c_3 \\ c_1 = -c_3 i \end{cases}$$

$\infty$  many sols.  $\Rightarrow \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$  not a basis

## ⑥ Linear Transformations

Ex  $T: \mathbb{C}^2 \rightarrow \mathbb{C}^2$   
 $T(\vec{v}) = A\vec{v}$

$$A = \begin{bmatrix} 0 & 1 \\ i & 2i \end{bmatrix}$$

a) Find the image of  $\vec{v} = \begin{bmatrix} 1+2i \\ 3 \end{bmatrix}$

$$A\vec{v} = \begin{bmatrix} 0 & 1 \\ i & 2i \end{bmatrix} \begin{bmatrix} 1+2i \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ -2+7i \end{bmatrix}$$

b) Find the preimage of  $\vec{w} = \begin{bmatrix} 4 \\ 7i \end{bmatrix}$

$$[A | \vec{w}]$$

$$\left[ \begin{array}{cc|c} 0 & 1 & 4 \\ i & 2i & 7i \end{array} \right]$$

$R_1 \leftrightarrow R_2$

$$\frac{1}{i} = -i$$

$$\left[ \begin{matrix} i & 2i & 7i \\ 0 & 1 & 4 \end{matrix} \right]$$

$$\frac{1}{i} R_1 \rightarrow R_1$$

$$\left[ \begin{matrix} 1 & 2 & 7 \\ 0 & 1 & 4 \end{matrix} \right]$$

$$R_1 - 2R_2 \rightarrow R_1$$

$$\left[ \begin{matrix} 1 & 0 & -1 \\ 0 & 1 & 4 \end{matrix} \right]$$

$$\begin{aligned} v_1 &= -1 \\ v_2 &= 4 \end{aligned}$$

$$\textcircled{\left( \begin{matrix} -1 \\ 4 \end{matrix} \right)}$$

components  
could be  
nonreal  
complex #s  
in principle

Anton Pd., S13

### F) $\vec{v} \cdot \vec{w}$

$$\vec{v} \cdot (\vec{v} \vec{w}) = \vec{v} (\vec{v} \cdot \vec{w})$$

$$\begin{aligned} &= v_1 \overline{w_1} + v_2 \overline{w_2} + \dots + v_n \overline{w_n} \quad (\text{Then, } \underline{\vec{v} \cdot \vec{v}} \stackrel{\text{real}}{\geq} 0) \\ &= \vec{v}^T \vec{w} \quad (\vec{v}, \vec{w} \text{ cols.}) \\ &= \|\vec{v}\|^2 \end{aligned}$$

$$\text{Ex } (1, i) \cdot (3, i) = (1)(3) + (i)(-i)$$

$$= 3 - i^2$$

$$= 8$$

If  $\vec{v}$  is in  $\mathbb{C}^n$ ,  $\|\vec{v}\| = |\vec{v}|$

$$= ④$$

$$\vec{v} \cdot \vec{w} = \overline{\vec{w} \cdot \vec{v}} \quad (\text{Complex inner product})$$

$$\|\vec{v}\| \triangleq (\vec{v} \cdot \vec{v})^{1/2}$$

## DIAGONALIZING COMPLEX MATRICES

8.5 (skip) Brings together Chs. 7, 8.

A sym. matrix is  
a real H matrix.

$$\begin{array}{l} A \text{ is Hermitian} \Leftrightarrow A = \bar{A}^T \\ P \text{ is unitary} \Leftrightarrow P^{-1} = \bar{P}^T \end{array} \quad \left| \begin{array}{l} \text{If } A \text{ real } (A = \bar{A}), A \text{ sym. } (A = A^T) \\ \text{If } P \text{ real } (P = \bar{P}), P \text{ OG } (P^{-1} = P^T) \end{array} \right.$$

Not the only ones  
that are!

↳ boils down to (real.) Sym.

Hermitian matrices are unitarily diag'e.  
OG'ly

Normal  $\Leftrightarrow U D' U^{-1}$

$$\begin{array}{l} A \bar{A}^T = \bar{A}^T A \\ \text{(H)} \quad \text{Both = I} \\ \text{Normal} \quad \text{for} \\ \text{unitary } A \\ N \quad H \quad D' \\ \text{Normal} \quad \text{unitary} \end{array}$$

$A$  Hermitian  $\Rightarrow$  there exists unitary  $P$ :

$$\bar{P}^T A P = \bar{P} \text{ diag}$$

Ch. 9: LP, Kantzig

Urban legend - turned in research problems as "homework"  
Berlin Airlift

American Airlines cut costs 10% in the 1980s  
90% of computing time in business ops.  
My friend - efficiency vs. employment