REVIEW

(1.1) Parametrize Sol'n Sets
Systems of Linear Eqs.
- Graphing
- Sub
- Addition

(1.2) Gaussian Elim.

System → Aug. Matrix ⇄ $\left[\begin{array}{cc|c} 1 & & \\ 0 & 1 & \end{array}\right]$ → System → Back-Sub

EROs: Row Interchange
- Rescaling
- Replacement

If $0\ 0 \cdots 0 \mid \neq 0 \rightarrow \phi$

Row-echelon form, which includes $\left[\begin{array}{cc|c} 1 & & \\ 0 & 1 & \end{array}\right]$

Gauss-Jordan Elim.

Uses RRE form

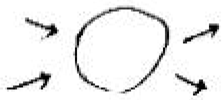
If $\left[\begin{array}{cc|c} 1 & 0 & \\ 0 & 1 & \end{array}\right]$ ← sol'n

If free vars, consistent ⇒
use this to parametrize the sol'n set
(∞ many sol'ns)

Homogeneous System: $\left[\begin{array}{c|c} & 0 \\ & \vdots \\ & 0 \end{array}\right]$ consistent

1.3 Poly Curve fitting

Network Analysis



flow in = flow out

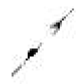
(2.1) Matrix Ops.

$A+B$, cA, AB (often $\neq BA$)

$[\ ]\ [\ ]$

Systems. $A\vec{x}=\vec{b}$ $\quad [A|\vec{b}]$

Ex $\rightarrow$ $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}}_{\text{"Initial pt."}} + t \underbrace{\begin{bmatrix} -5 \\ 3 \\ 1 \end{bmatrix}}_{\text{"Direction"}}$



(2.2) Matrix Eqs.

Matrix Mult. Props.

O, I

A^T, Props.

A sym. $\leftrightarrow A=A^T$

(2.3) A^{-1}

$AA^{-1}=A^{-1}A=I$

Find A^{-1}

2×2 Shortcut

$[A|I]$

$\downarrow$

$[I|A^{-1}]$

Props.

Solve $A\vec{x}=\vec{b}$ (A inv'e) $\Rightarrow \vec{x}=A^{-1}\vec{b}$

2.4 Elementary Matrices

EA

↑ performs ERO
has E^{-1}

$A = \underset{\triangle}{L}\,\underset{\nabla}{U}$?

Solve $A\vec{x} = \vec{b}$

Solve $L\vec{y} = \vec{b}$ $[L \mid \vec{b}]$ $\xrightarrow{\text{sol'n}}$

Solve $U\vec{x} = \vec{y}$ $[U \mid \vec{y}]$ → sol'n $= \vec{x}$

forward 1st

Invertible Matrix Thm.

2.5 Least Squares

3.1 $|A|$

2×2 butterfly $(ad-bc)$
3×3 Sarrus
In general, can expand by cofactors
  Exploit 0s
  Sign matrix

$|{}_0\nabla|$, $|\Delta^0|$ = product "along" main diag.

3.2 Use EROs, ECOs $\rightarrow |{}_0\nabla|, |\Delta^0|$
  switch $|A| = -|B|$
  replacement $|A| = |B|$
  factor out of a row/col.

3.3 Props. of Dets.
Proof by Induction

3.4 Cramer's Rule
(Geometry)

(4.1) Vectors in $\mathbb{R}^n$

$+$ 

$c\vec{v} \parallel \vec{v}$ 

$c\vec{v} = \vec{0} \rightarrow c = 0$ or $\vec{v} = \vec{0}$

(4.2) Vector Spaces

10 props.
Define "$+$", "$c\vec{v}$"
in $\mathbb{R}$

<u>Ex</u> $\mathbb{R}^n$, $M_{m,n}$, P, P_n

(4.3) Subspaces

$W \neq \emptyset$
$W \subseteq V$ (VS)
W is a VS
show closure under "$+$", "$c\vec{v}$"

Transitive 

$W_1 \cap W_2$ 

Subspaces of $\mathbb{R}^n$
$\mathbb{R}^2$ 

4.4 Linear Combos

$\vec{w} = c_1\vec{v}_1 + \dots + c_k\vec{v}_k$

In $\mathbb{R}^m$, solve $[\vec{v}_1 \cdots \vec{v}_k]\begin{bmatrix} c_1 \\ \vdots \\ c_k \end{bmatrix} = \vec{w}$

Span$(\{\vec{v}_1, \dots, \vec{v}_k\})$ = set of all LCs of "$\vec{v}_i$"s
is a subspace

In $\mathbb{R}^m$, if $[\vec{v}_1 \cdots \vec{v}_k] \to$ [row-ech. shape] $\to$ each row has PP
then span $\mathbb{R}^m$

Correspondences

$M_{m,n} \cong \mathbb{R}^{mn}$

$P_n \cong \mathbb{R}^{n+1}$

LI

If $c_1\vec{v}_1 + \dots + c_k\vec{v}_k = \vec{0}$, then all $c_i = 0$

In $\mathbb{R}^m$, solve $\underbrace{[\vec{v}_1 \cdots \vec{v}_k]}_{\text{[row-ech shape]} \to \text{each col. has PP}}\begin{bmatrix} c_1 \\ c_k \end{bmatrix} = \vec{0} \Rightarrow \vec{c} = \vec{0}$ only

Each "$\vec{v}_i$" valuable.

$\vec{0} \Rightarrow$ LD

4.5 Basis

$B = \{\vec{b}_1, \ldots, \vec{b}_m\}$

span V
LI
$m = \dim(V)$

In $\mathbb{R}^m$, $[\vec{b}_1 \cdots \vec{b}_m] \sim \begin{bmatrix} PP & \cdots & \\ 0 & \cdots & PP \end{bmatrix}$

Standard Bases for
$\mathbb{R}^m$, $M_{m,n}$, P_n

$\vec{w}$ has unique coords. rel. to B (ordered)
Solve $[\vec{b}_1 \cdots \vec{b}_m]\vec{c} = \vec{w}$

4.6 $A_{m \times n}$ $\begin{array}{c|c} & 1 \quad n \\ m & [\quad] \end{array}$
$Row(A)$ = Span (row vectors) a subspace of $\mathbb{R}^n$
$Col(A)$ col $\mathbb{R}^m$
$rank(A)$ = dim of both
= # PPs in $\begin{bmatrix} \text{row-ech.} \\ \text{shape} \end{bmatrix}$

Find basis for
$Row(A)$
$[A] \sim \begin{bmatrix} \text{row-ech} \\ \text{shape} \end{bmatrix}$ $\Rightarrow$ Grab pivot rows
$Col(A)$
$([A] \sim \begin{bmatrix} \text{row-ech} \\ \text{shape} \end{bmatrix})$
Identify pivot cols
Grab corresp. cols here!
or $Col(A) = Row(A^T)$

Analyze Span $(\{\vec{v}_1, \ldots, \vec{v}_k\})$

$$\underbrace{\begin{bmatrix} - \vec{v}_1 - \\ \vdots \\ - \vec{v}_k - \end{bmatrix}}_{A}$$

Analyze Row(A)

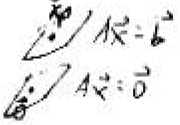
$N(A)$ = nullspace of A
$= \{\vec{x} \mid A\vec{x} = \vec{0}\}$
= a subspace of $\mathbb{R}^n$

nullity(A) = dim$(N(A))$

Find basis:
Write sol'n set of $A\vec{x} = \vec{0}$ in param. form.
Grab "coeff. vectors"

$$\underbrace{\text{rank}(A)}_{\#\text{ pivot cols}} + \underbrace{\text{nullity}(A)}_{\#\text{ nonpivot cols}} = \underbrace{n}_{\#\text{ cols.}}$$

Systems
If $\vec{x}_p$ solves $A\vec{x} = \vec{b}$,
sol'n set $= \{\vec{x}_p + \vec{x}_h \mid \vec{x}_h$ solves $A\vec{x} = \vec{0}\}$



$A\vec{x} = \vec{b}$ is consistent $\leftrightarrow$ $\vec{b}$ is in Col(A)

Ch.5 $\mathbb{R}^n$

5.1 $\|\vec{v}\|$

$\vec{u} = \frac{1}{\|\vec{v}\|}\vec{v}$ 

normalizing

$d(\vec{v},\vec{w}) = \|\vec{v}-\vec{w}\|$

$\vec{v}\cdot\vec{w}$ Ex. of inner product $\langle \vec{v},\vec{w}\rangle$

If cols., $\vec{v}^T\vec{w}$

$= \#$

Props.

 $\cos\theta = \frac{\vec{v}\cdot\vec{w}}{\|\vec{v}\|\|\vec{w}\|}$

in $[-1,1]$ by Cauchy-Schwarz Ineq.

$\vec{v}\perp\vec{w} \leftrightarrow \vec{v}\cdot\vec{w}=0$

orthogonal

Triangle Ineq.

Pyth. Thm.

5.2 $\text{proj}_{\vec{w}} \vec{v} = \left(\dfrac{\vec{v}\cdot\vec{w}}{\vec{w}\cdot\vec{w}}\right)\vec{w}$

5.3

OG set : pairwise OG

(LI if none $= \vec{0}$)

ON set : OG, unit vecs.

Any OG set of n vecs (none $\vec{0}$) is a basis for $\mathbb{R}^n$

If $B = \{\vec{u}_1, \dots, \vec{u}_n\}$ is an ON basis for $\mathbb{R}^n$

$$\vec{w} = \underbrace{c_1\vec{u}_1}_{\text{proj}_{\vec{u}_1}\vec{w}} + \dots + \underbrace{c_n\vec{u}_n}_{\text{proj}_{\vec{u}_n}\vec{w}}$$

$c_1 = \vec{w}\cdot\vec{u}_1 \quad c_n = \vec{w}\cdot\vec{u}_n$ ← coords wrt B

Gram-Schmidt

Basis for $\mathbb{R}^n$ (or a subspace)

$\{\vec{v}_1, \dots, \vec{v}_n\}$

↓

OG : $\vec{w}_1 = \vec{v}_1$

$\vec{w}_2 = \vec{v}_2 - \text{proj}_{\vec{w}_1}\vec{v}_2$

$\vec{w}_3 = \vec{v}_3 - \text{proj}_{\vec{w}_1}\vec{v}_3 - \text{proj}_{\vec{w}_2}\vec{v}_3$

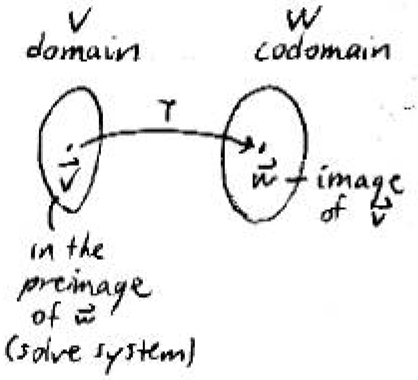
etc.

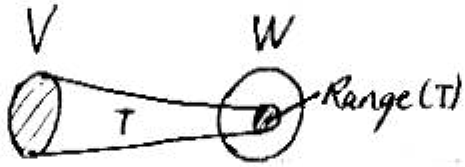
↓

ON : normalize →

$\{\vec{u}_1, \dots, \vec{u}_n\}$

6.1 $T(\vec{v}) = \vec{w}$





LTs

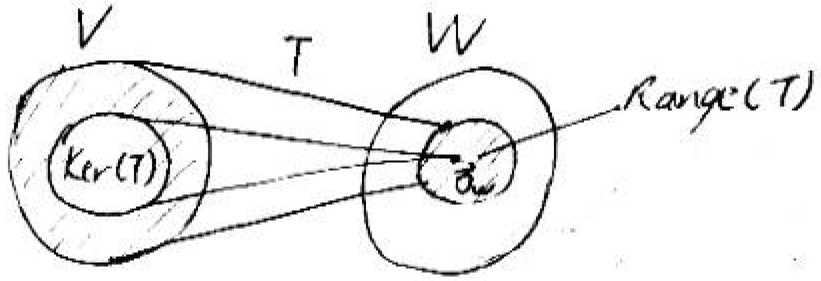
$$T(\vec{v}_1 + \vec{v}_2) = T(\vec{v}_1) + T(\vec{v}_2)$$
$$T(c\vec{v}_1) = cT(\vec{v}_1)$$

Know how T maps a basis of $V \to$ Know T

$T(\vec{v}) = \underset{(m \times n)}{A}\vec{v}$ defines linear $T: \mathbb{R}^n \to \mathbb{R}^m$

0, identity, proj., rotation

6.2



T is 1-1
(NO ⇉·)
$\leftrightarrow Ker(T) = \{\vec{0}_V\}$

T is onto

$\leftrightarrow Range(T) = W$

Isomorphism: 1-1 and onto
$V \cong W \leftrightarrow dim(V) = dim(W) \quad (<\infty)$

$\underline{T(\vec{x}) = A\vec{x}}$ $\quad A_{m\times n} \quad \overset{\mathbb{R}^n}{O} \to \overset{\mathbb{R}^m}{O}$

$Ker(T) = N(A) \xrightarrow{dim} nullity(T) = \#$ nonpivot cols in A
$Range(T) = Col(A) \to rank(T) = \#$pivot
$\Big\}$ sum $= n = dim(V)$

T is 1-1 $\leftrightarrow A \sim [\quad]$
(P) in each col.
$rank(T) = n$

T is onto $\leftrightarrow$ (P) in each row

6.3 Linear $T: \mathbb{R}^n \to \mathbb{R}^m$

Standard matrix:

$$A = \begin{array}{cc} & n \\ m & \left[\underbrace{T(\vec{e}_1) \cdots T(\vec{e}_n)}_{\text{images of st. basis vectors}} \right] \end{array}$$

(Shortcut: coeffs, 0-cols)

$\Rightarrow T(\vec{v}) = A\vec{v}$ for <u>every</u> $\vec{v}$ in $\mathbb{R}^n$

$T_2 \circ T_1(\vec{v}) = T_2(T_1(\vec{v})) = A_2 A_1 \vec{v}$

$T^{-1}(\vec{v}) = A^{-1}\vec{v}$

T inv'e $\leftrightarrow$ T isom

6.4 $\underbrace{A' = P^{-1} A}_{\text{similar}} P$

$A \overset{s}{\sim} A$

$A \overset{s}{\sim} A' \to A' \overset{s}{\sim} A$

$A \overset{s}{\sim} A',\ A' \overset{s}{\sim} A'' \to A \overset{s}{\sim} A''$

Transition matrix P from $B' = \{\vec{v}_1, \ldots, \vec{v}_n\}$
to $B = \{\vec{e}_1, \ldots, \vec{e}_n\}$ (standard)

$P = [\vec{v}_1 \cdots \vec{v}_n]$

$P[\vec{x}]_{B'} = [\vec{x}]_B$

$[\vec{x}]_{B'} = \underbrace{P^{-1}}_{B \to B'}[\vec{x}]_B$

Matrix for T rel. to B'

$$A' = \underbrace{P^{-1}}_{B \to B'} \; \underbrace{A}_{\substack{\text{st.} \\ \text{matrix} \\ (\text{rel. to } B)}} \; \underbrace{P}_{B' \to B}$$

$$A \sim A'$$

7.1

$A\vec{x} = \lambda \vec{x}$

eval evec ($\neq \vec{0}$)

E_λ is a subspace of $\mathbb{R}^n$.

$1 \le \dim(E_\lambda) \le$ multiplicity of λ

Find evals

Solve $\underbrace{\underbrace{\det(\lambda I - A)}_{\text{char poly of } A} = 0}_{\text{char. eq.}}$ for λ

Find evecs for λ

Solve $(\lambda I - A)\vec{x} = \vec{0}$ $(\vec{x} \neq \vec{0})$

evals evals

alg. vs. geom. multiplicity of λ_i

7.2

A diag'e $\iff$ there exists P: $P^{-1} A \underbrace{P = D}_{\text{same evals}}$ (some diag. matrix)

$\iff$ there exist n LI evecs

$P = [\vec{p}_1 \cdots \vec{p}_n]$

$D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$

If $\{\underbrace{\vec{x}_1, \ldots, \vec{x}_k}_{\text{evecs, each from diff. espaces}}\} \Rightarrow LI$

Find a basis B' s.t. matrix for T is diag.

7.3

A sym. (real) $\Rightarrow$ A diag'e, all evals real

P is OG (cols are ON)

$P^{-1} = P^{T}$

$PP^{T} = I$

A is orthogonally diag'e $\leftrightarrow$ A sym.

$$\underbrace{P^{-1}}_{=P^{T}} A P = D$$

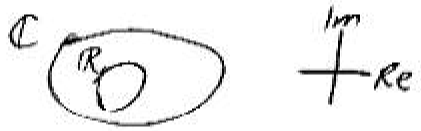
is OG

Normalize cols!

8.1

$i=\sqrt{-1}$, $i^2=-1$

$\sqrt{-18} = i\sqrt{18} = 3i\sqrt{2}$



Simplify $\sqrt{\ }$s before $+, -, \times, \div$
Quadratic Formula
Matrix Ops.

8.2

$\bar{z}$
$|z|$, Props.
$\div$ (can $\cdot\frac{i}{i}$ or $\cdot\frac{\bar{z}}{\bar{z}}$)
A^{-1}

8.4

Complex VSs
set of scalars $= \mathbb{C}$
$\mathbb{C}^n$, bases
LTs

$\vec{v}\cdot\vec{w} = \underbrace{\vec{v}^T\overline{\vec{w}}}_{\text{cols}}$

$\|\vec{v}\| = \sqrt{\vec{v}\cdot\vec{v}} = \sqrt{\Sigma|v_i|^2}$